

$$T = 6, \quad \omega_0 = 2\pi/6 = \pi/3$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{6} \int_{-1}^5 x(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{6} \int_{-1}^0 2 e^{-jn\omega_0 t} dt + \frac{1}{6} \int_2^3 2 e^{-jn\omega_0 t} dt$$

$$= \frac{1}{3} \left[ \frac{-1}{jn\omega_0} e^{-jn\omega_0 t} \Big|_{-1}^0 + \frac{1}{3} \frac{-1}{jn\omega_0} e^{-jn\omega_0 t} \Big|_2^3 \right], n \neq 0$$

$$= \frac{1}{jn\pi} (1 - e^{jn\pi/3}) - \frac{1}{jn\pi} (e^{-jn\pi} - e^{-jn\pi/3}), n \neq 0$$

$$= \frac{1}{jn\pi} \left[ e^{jn\pi/6} (e^{-jn\pi/6} - e^{jn\pi/6}) - e^{-jn\pi/6} (e^{jn\pi/6} - e^{-jn\pi/6}) \right]$$

$$= \frac{2}{n\pi} \left[ e^{jn\pi/6} \sin\left(\frac{n\pi}{6}\right) + e^{-jn\pi/6} \sin\left(\frac{n\pi}{6}\right) \right]$$

$$= \frac{2}{n\pi} \sin\left(\frac{n\pi}{6}\right) \left[ -e^{jn\pi/6} + e^{-jn\pi/6} \right] e^{-\frac{2\pi}{6}nj}$$

$$c_n = \frac{-4j}{n\pi} \sin\left(\frac{n\pi}{6}\right) \sin\left(\frac{n\pi}{2}\right) e^{-jn\pi/3}$$

if  $n=0$ :

$$C_0 = \frac{1}{6} \int_{-1}^0 -2 dt + \frac{1}{6} \int_0^3 2 dt = 0$$

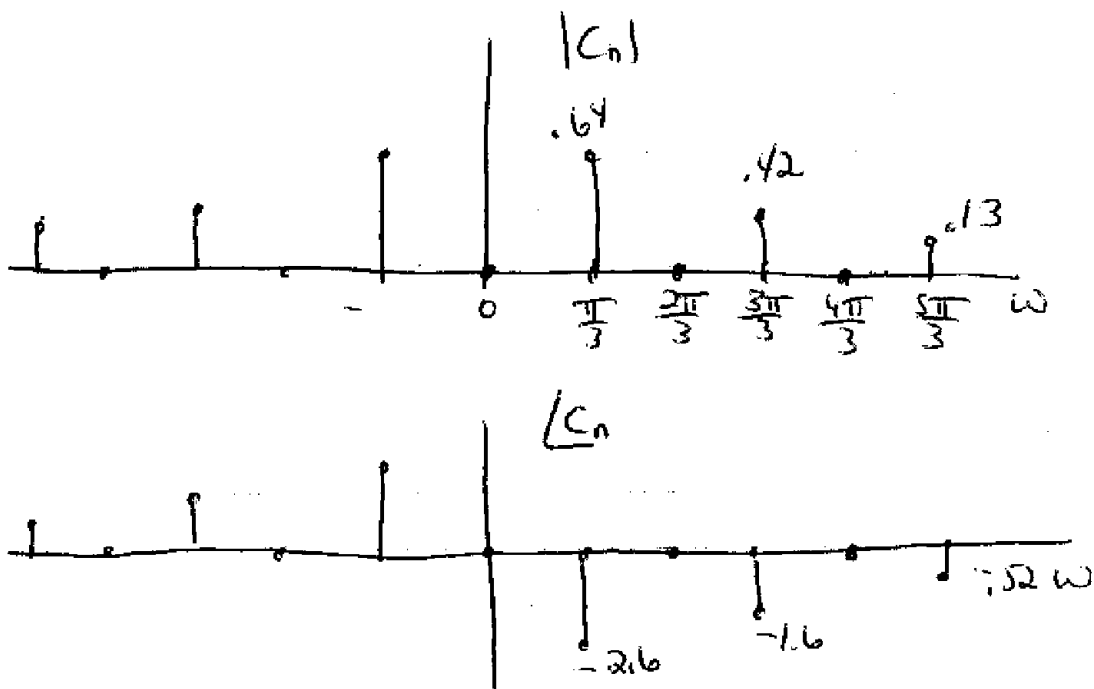
$$C_1 = -.5513 - .318j = 0.64 e^{j2.6}, C_{-1} = \bar{C}_1$$

$$C_2 = 0$$

$$C_3 = 0 - j.42 = 0.42 e^{j1.6}, C_{-3} = \bar{C}_3$$

$$C_4 = 0$$

$$C_5 = 0.1103 - .0637j = 0.127 e^{j.52}, C_{-5} = \bar{C}_5$$

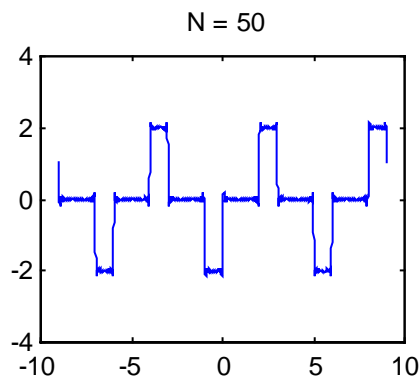
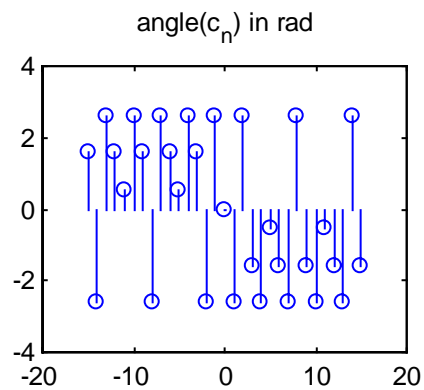
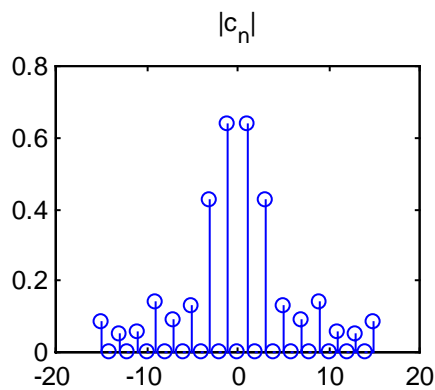


Some of the work can be done using MATLAB:

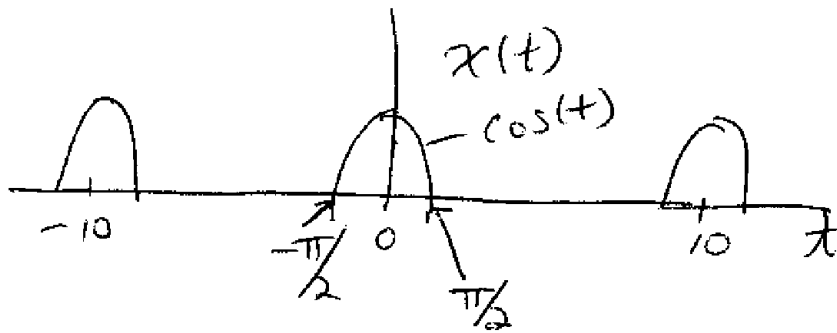
```
» n=1:15;
» cn=-4*j./n/pi.*sin(pi*n/6).*sin(n*pi/2).*exp(-j*n*pi/3);
» n=-15:-1;
» c_n=-4*j./n/pi.*sin(pi*n/6).*sin(n*pi/2).*exp(-j*n*pi/3);
» cn=[c_n 0 cn];
» n=-15:15;
» subplot(221),stem(n,abs(cn))
» title('|c_n|')
» subplot(222),stem(n,angle(cn))
» title('angle(c_n) in rad')
```

To plot the Fourier series to check your answers:

```
T=6;
w0 = 2*pi/T;
t = -1.5*T:T/1000:1.5*T;
N = input('Number of harmonics ');
c0 = 0;
x = c0*ones(1,length(t)); % dc component
for n=1:N,
    cn = -4*j/n/pi*sin(pi*n/6)*sin(n*pi/2)*exp(-j*n*pi/3);
    c_n = conj(cn);
    x = x + cn*exp(j*n*w0*t) + c_n*exp(-j*n*w0*t);
end
plot(t,x)
title([' N = ',num2str(N)])
```



2.



$$T=10 \Rightarrow \omega_0 = 2\pi/10 = \pi/5$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T} \int x(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{10} \int_{-\pi/2}^{\pi/2} \cos(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{20} \int_{-\pi/2}^{\pi/2} (e^{jt} + e^{-jt}) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{20} \int_{-\pi/2}^{\pi/2} e^{jt(1-n\omega_0)} + e^{-jt(1+n\omega_0)} dt$$

$$= \frac{1}{20} \frac{1}{j(1-n\omega_0)} \left[ e^{j\frac{\pi}{2}(1-n\omega_0)} - e^{-j\frac{\pi}{2}(1-n\omega_0)} \right]$$

$$+ \frac{-1}{20j(1+n\omega_0)} \left[ e^{-j\frac{\pi}{2}(1+n\omega_0)} - e^{j\frac{\pi}{2}(1+n\omega_0)} \right]$$

$$= \frac{1}{20j} \frac{1}{(1-n\omega_0)} \left[ j e^{-j\frac{\pi}{2}n\omega_0} + j e^{j\frac{\pi}{2}n\omega_0} \right]$$

$$- \frac{1}{20j} \frac{1}{(1+n\omega_0)} \left[ j e^{-j\frac{\pi}{2}n\omega_0} - j e^{j\frac{\pi}{2}n\omega_0} \right]$$

$$= \frac{1}{10(1-n\omega_0)} \cos\left(\frac{\pi}{2} n\omega_0\right) + \frac{1}{10(1+n\omega_0)} \cos\left(\frac{\pi}{2} n\omega_0\right)$$

$$= \cos\left(\frac{\pi}{2} n\omega_0\right) \frac{1}{10} \left[ \frac{1}{1-n\omega_0} + \frac{1}{1+n\omega_0} \right]$$

$$= \cos\left(\frac{\pi}{2} n\omega_0\right) \frac{1}{10} \frac{2}{1-(n\omega_0)^2}$$

$$C_n = \cos\left(\frac{\pi}{2} n\omega_0\right) \frac{1}{5(1-(n\omega_0)^2)}$$

$$\text{for } \omega_0 = \pi/5 : C_n = \cos\left(\frac{\pi^2 n}{10}\right) \frac{1}{5(1-(\frac{n\pi}{5})^2)}$$

$$C_0 = 1/5$$

$$C_1 = \cos\left(\frac{\pi^2}{10}\right) \frac{1}{5(1-(\frac{\pi}{5})^2)} = 0.1822$$

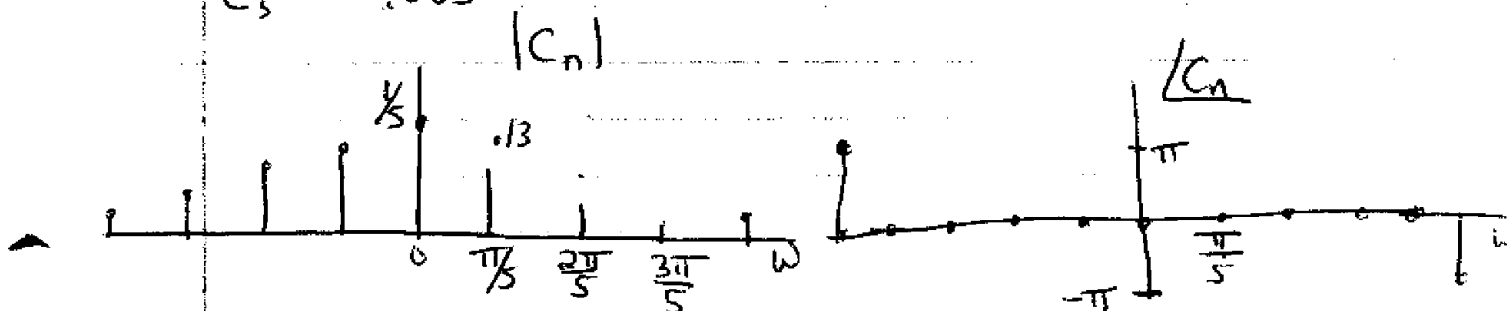
$$C_2 = 0.1355$$

$$C_3 = 0.0771$$

$$C_4 = 0.0266$$

$$C_5 = -0.005$$

$$C_{-n} = \overline{C_n} = C_n$$



Some of the work can be done using MATLAB:

```

» n=-10:10;
» cn=cos(pi/2*n*w0)/5./(1-(n*w0).^2);
» subplot(221),stem(n,abs(cn))
» title('|c_n|')
» subplot(222),stem(n,angle(cn))
» title('angle(c_n) in rad')

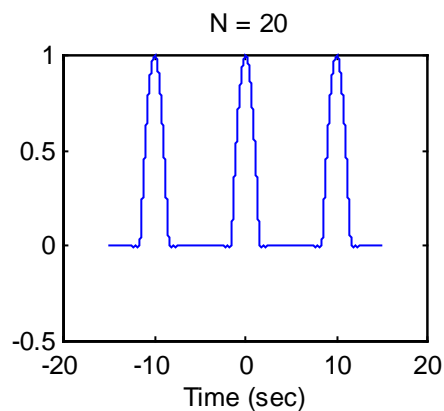
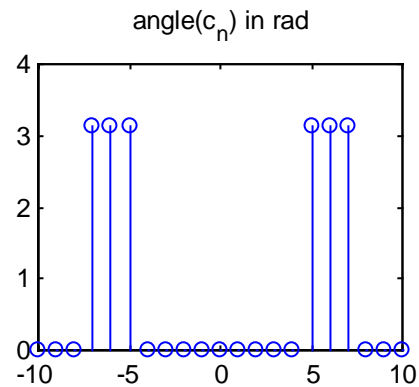
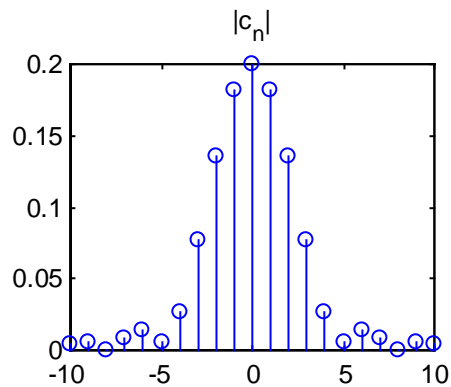
```

will give first plots shown below. To check your answer, you can plot the truncated series and see if it converges correctly.

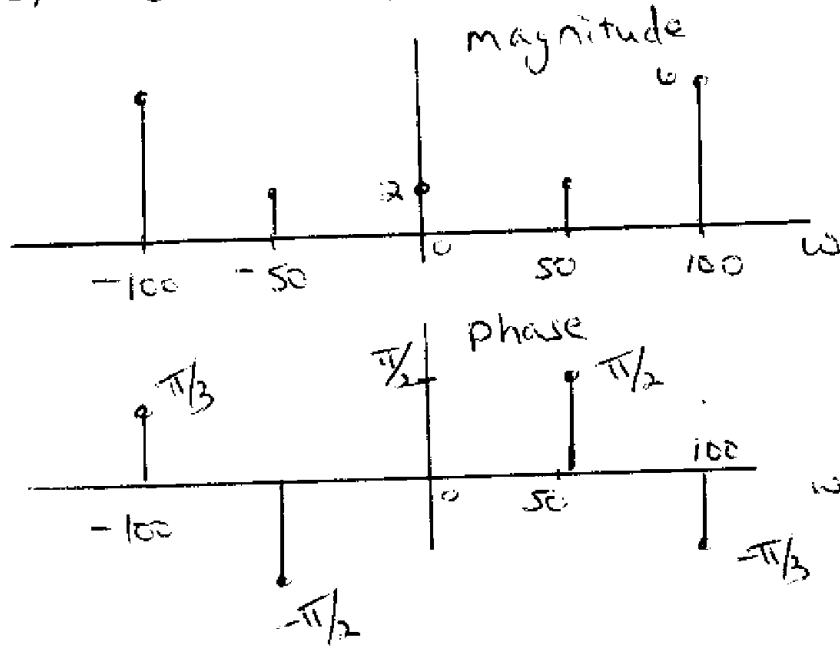
```

T=10;
w0 = 2*pi/T;
t = -1.5:T/1000:1.5*T;
N = input('Number of harmonics ');
c0 = 1/5;
x = c0*ones(1,length(t)); % dc component
for n=1:N,
    cn = cos(pi/2*n*w0)/5/(1-(n*w0)^2);
    c_n = cn;
    x = x + cn*exp(j*n*w0*t) + c_n*exp(-j*n*w0*t);
end
plot(t,x)
title([' N = ',num2str(N)])

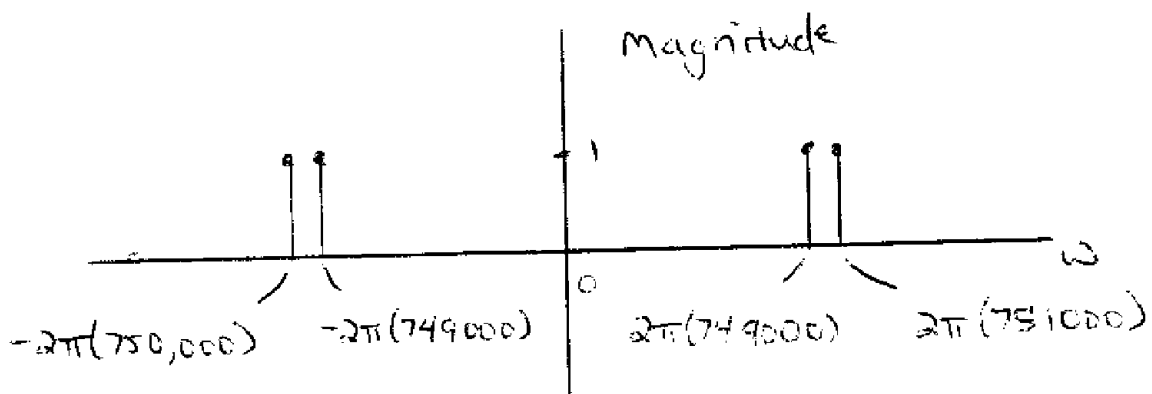
```



$$3a) \quad x(t) = 2 + 4 \cos(50t + \pi/2) + 12 \cos(100t - \pi/3)$$



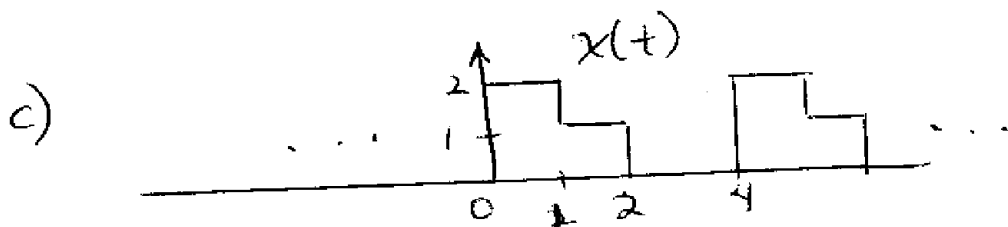
$$\begin{aligned}
 b) \quad x(t) &= 4 \cos(2\pi(1000)t) \cos(2\pi(75000)t) \\
 &= 4 \left[ \frac{e^{j2\pi(1000)t} + e^{-j2\pi(1000)t}}{2} \right] \left[ \frac{e^{j2\pi(75000)t} + e^{-j2\pi(75000)t}}{2} \right] \\
 &= e^{-j2\pi(749000)t} + e^{-j2\pi(749000)t} + e^{j2\pi(751000)t} + e^{j2\pi(751000)t}
 \end{aligned}$$



phase is zero

$\omega_0$  is greatest common factor,

$$\omega_0 = 2\pi(1000)$$



$$T=4 \quad \text{so} \quad \omega_0 = 2\pi/4 = \pi/2$$

$$C_0 = \frac{1}{4} \int_0^2 x(t) dt = 3/4$$

$$C_n = \frac{1}{4} \int_0^2 x(t) e^{-jn\pi/2 t} dt$$

$$= \frac{1}{4} \int_0^1 2 e^{-jn\pi/2 t} dt + \frac{1}{4} \int_1^2 e^{-jn\pi/2 t} dt$$

$$= \frac{1}{2} \frac{1}{-jn\pi/2} e^{-jn\pi/2 t} \Big|_0^1 + \frac{1}{4} \frac{1}{-jn\pi/2} e^{-jn\pi/2 t} \Big|_1^2$$

$$= \frac{1}{-jn\pi} [e^{-jn\pi/2} - 1] + \frac{1}{-jn\pi} [e^{-jn\pi} - e^{-jn\pi/2}]$$

This can be simplified

$$e^{-jn\pi} = (-1)^n$$

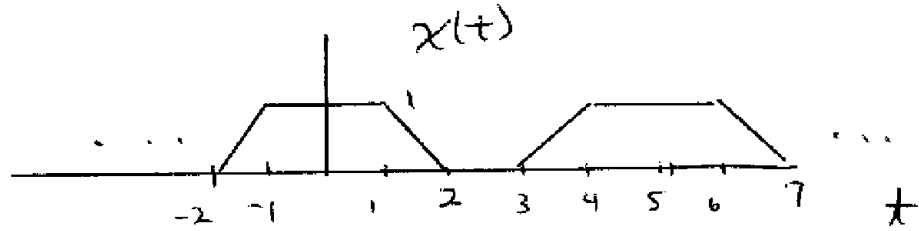
$$C_n = \frac{1}{jn\pi} \left[ 1 - \frac{1}{2}(-1)^n - \frac{1}{2}e^{-jn\pi/2} \right]$$

$$C_0 = 3/4$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\pi/2 t}$$



d)



$$T = 5 \quad \omega_0 = \frac{2\pi}{5}$$

$$C_0 = \frac{1}{5} \int_{-2}^2 x(t) dt = 3/5$$

$$C_n = \frac{1}{5} \int_{-2}^{-1} (t+2) e^{-jn\omega_0 t} dt + \frac{1}{5} \int_{-1}^1 e^{-jn\omega_0 t} dt + \frac{1}{5} \int_1^2 (-t+2) e^{-jn\omega_0 t} dt$$

recall

$$\int t e^{at} dt = \frac{e^{at}}{a^2} (at-1)$$

$$C_n = \frac{1}{5} \left[ \frac{e^{-jn\omega_0 t}}{(-jn\omega_0)^2} (-jn\omega_0 t - 1) - \frac{2}{jn\omega_0} e^{-jn\omega_0 t} \right]_{-2}^{-1}$$

$$+ \frac{-1}{jn\omega_0} e^{-jn\omega_0 t} \Big|_{-1}^1 + \frac{-e^{-jn\omega_0 t}}{(-jn\omega_0)^2} (-jn\omega_0 t - 1)$$

$$- \frac{2}{jn\omega_0} e^{-jn\omega_0 t} \Big|_1^2$$

$$\begin{aligned}
C_n = & \frac{1}{5} \left[ \frac{e^{jn\omega_0}}{-n^2\omega_0^2} (jn\omega_0 - 1) - \frac{2}{jn\omega_0} e^{jn\omega_0} + \frac{e^{jn\omega_0 2}}{\omega_0^2 n^2} (jn\omega_0 2 - 1) \right. \\
& + \frac{2}{jn\omega_0} e^{jn\omega_0 2} - \frac{1}{jn\omega_0} e^{-jn\omega_0} + \frac{1}{jn\omega_0} e^{jn\omega_0} \\
& + \frac{e^{jn\omega_0 2}}{n^2\omega_0^2} (-jn\omega_0 2 - 1) - \frac{2}{jn\omega_0} e^{-jn\omega_0 2} - \frac{e^{-jn\omega_0}}{n^2\omega_0^2} (jn\omega_0 - 1) \\
& \left. + \frac{2}{jn\omega_0} e^{-jn\omega_0} \right]
\end{aligned}$$

$$\begin{aligned}
= & \frac{1}{5} e^{jn\omega_0} \left( \frac{-j}{n\omega_0} + \frac{1}{n^2\omega_0^2} + \frac{2j}{n\omega_0} - \frac{j}{\omega_0 n} \right) \\
& + \frac{1}{5} e^{-jn\omega_0} \left( \frac{j}{n\omega_0} + \frac{j}{n\omega_0} + \frac{1}{n^2\omega_0^2} - \frac{2j}{n\omega_0} \right) \\
& + \frac{1}{5} e^{j2n\omega_0} \left( \frac{j2}{\omega_0^2 n^2} - \frac{1}{\omega_0^2 n^2} - \frac{2j}{n\omega_0} \right) \\
& + \frac{1}{5} e^{-j2n\omega_0} \left( \frac{-j2}{n\omega_0} - \frac{1}{n^2\omega_0^2} + \frac{2j}{n\omega_0} \right)
\end{aligned}$$

$$= \frac{1}{5n^2\omega_0^2} (e^{jn\omega_0} + e^{-jn\omega_0}) - \frac{1}{5n^2\omega_0^2} (e^{j2n\omega_0} + e^{-j2n\omega_0})$$

$$C_n = \frac{2}{5n^2\omega_0^2} [\cos(n\omega_0) - \cos(2n\omega_0)], \quad C_0 = 3/5$$

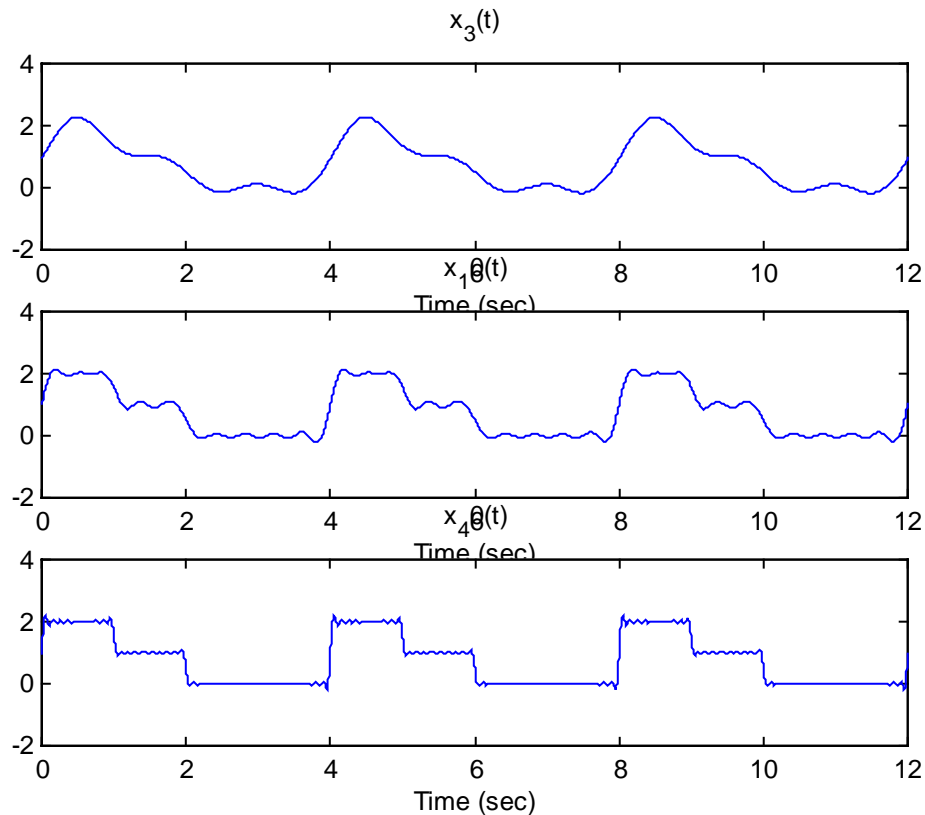
$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

```

% truncated Fourier series for a staircase signal
T = 4;
w0 = 2*pi/T;
t = 0:T/1000:3*T;
N = input('input N')
nneg = -N:-1;
npos = 1:N;
c0 = 3/4;
cneg = 1/j/pi./nneg.*(1-.5*(-1).^nneg-.5*exp(-j*nneg*pi/2));
cpos = 1/j/pi./npos.*(1-.5*(-1).^npos-.5*exp(-j*npos*pi/2));
n = [nneg 0 npos];
c_n = [cneg c0 cpos];
x = c_n*exp(j*w0*n'*t);
x = real(x); % strips off negligible imaginary parts
plot(t,x)
xlabel('Time (sec)')
title(['x',num2str(N),'(t)'])

```

The truncated series is plotted from top to bottom for  $N = 3$ ,  $N = 10$ , and  $N = 40$ .

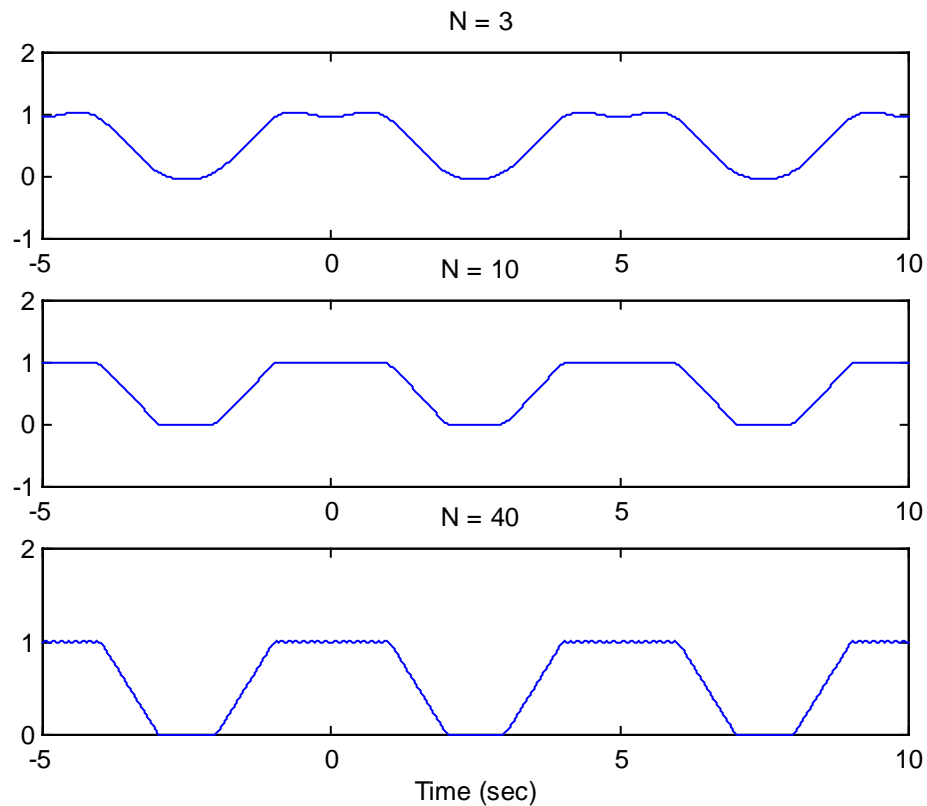


This program uses a loop to compute the Fourier series rather than the large matrix multiplication. One student had difficulty using the student version of Matlab with the other program (it took too much memory). This one is less efficient because of the loop, but it uses less memory.

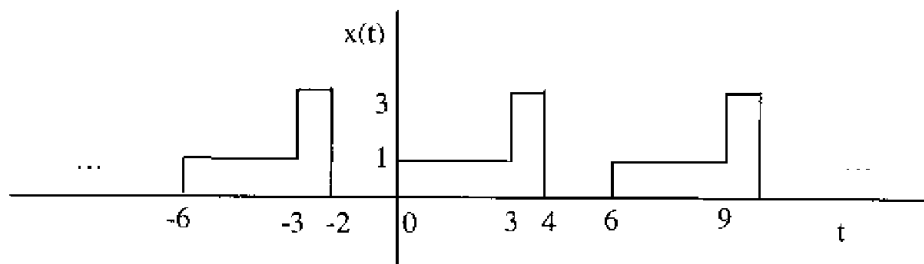
```
% computes exponential Fourier series for trapezoidal wave
```

```
T = 5;
w = 2*pi/5;
t = -T:T/1000:2*T;
N = input('Number of harmonics ');
c0 = 3/5;
w0 = pi;
xN = c0*ones(1,length(t)); % dc component
for n=1:N,
    cn = 2/5./n.^2/w/w*(cos(n*w)-cos(2*n*w));
    c_n = cn;
    xN = xN + cn*exp(j*n*w*t) + c_n*exp(-j*n*w*t);
end
plot(t,xN)
title([' N = ',num2str(N)])
```

Running this program for N = 3, N= 10 and N= 40 gives:



5.



$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$T=6, \omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$$

$$C_n = \frac{1}{6} \int_0^6 x(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{6} \int_0^3 e^{-jn\frac{\pi}{3}t} dt + \frac{1}{6} \int_3^4 3 e^{-jn\frac{\pi}{3}t} dt$$

$$= \frac{1}{6} \left. \frac{-1}{jn\frac{\pi}{3}} e^{-jn\frac{\pi}{3}t} \right|_0^3 + \frac{1}{2} \left. \frac{-1}{jn\frac{\pi}{3}} e^{-jn\frac{\pi}{3}t} \right|_3^4, n \neq 0$$

$$C_n = \frac{-1}{j2\pi n} (e^{-jn\pi} - 1) - \frac{3}{j2\pi n} (e^{-jn\frac{4\pi}{3}} - e^{-jn\pi}), n \neq 0$$

if  $n=0$ ,  $C_0 = \frac{1}{6} \int_0^6 x(t) dt = 1$

approximation has error due to Gibbs phenomenon

