1 a) $x[n] = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$= \tfrac{1}{4}\left(1 + e^{-j\Omega} + e^{-j2\Omega} + e^{-j3\Omega}\right)$$

$$= \tfrac{1}{4} e^{-j\frac{3\Omega}{2}} \left(e^{+j\frac{3\Omega}{2}} + e^{j\frac{\Omega}{2}} + e^{-j\frac{\Omega}{2}} + e^{-j\frac{3\Omega}{2}}\right)$$

$$= \tfrac{1}{2} e^{-j\frac{3\Omega}{2}} \left(\cos\left(\tfrac{3\Omega}{2}\right) + \cos\left(\tfrac{\Omega}{2}\right)\right)$$
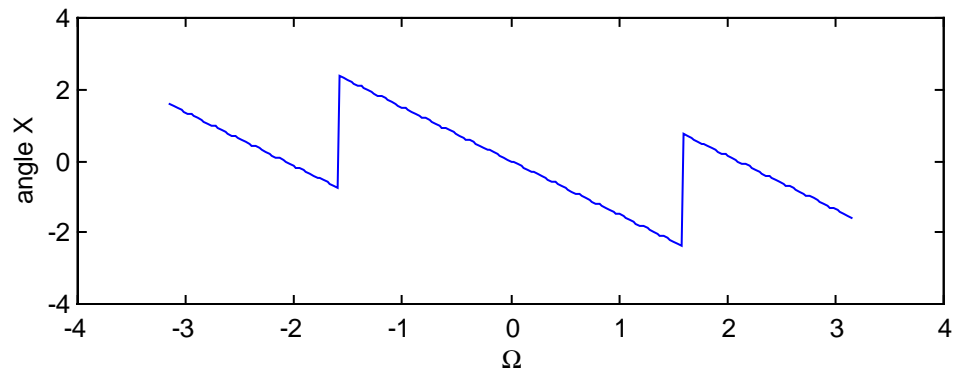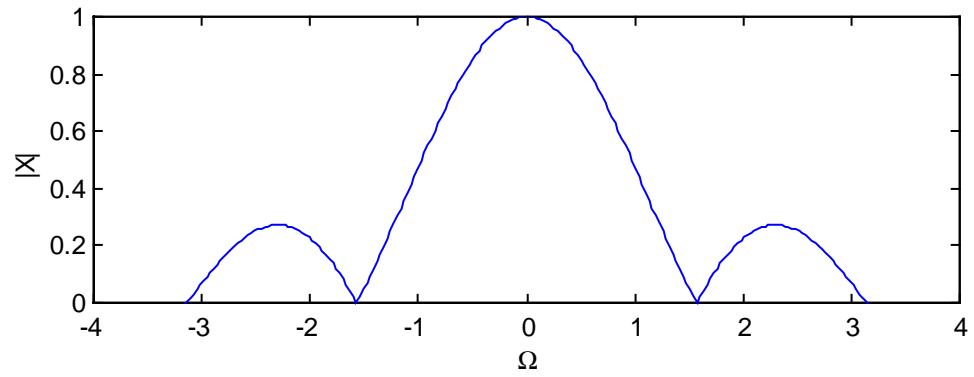
plot shown later

b) $x[n] = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$

$$X(\Omega) = 1 - 2e^{-j\Omega} + e^{-j2\Omega}$$

$$= e^{-j\Omega}\left(e^{j\Omega} - 2 + e^{-j\Omega}\right)$$

$$= e^{-j\Omega}\, 2\left(\cos(\Omega) - 1\right)$$

$$= e^{-j(\Omega \pm \pi)}\, 2\left(1 - \cos(\Omega)\right)$$
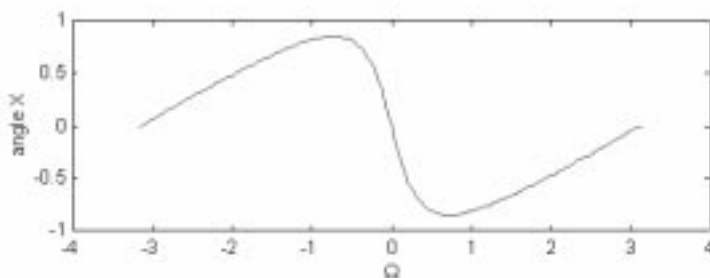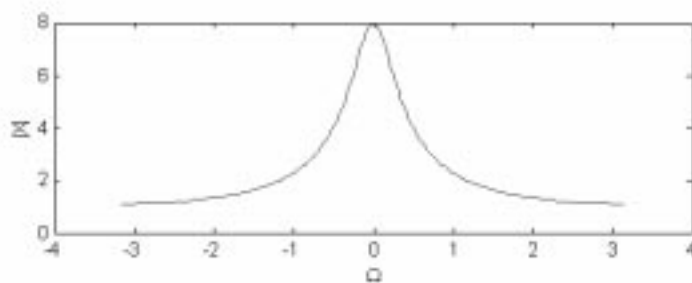
1. a) plot

c) $x[n] = 2\left(\frac{3}{4}\right)^n u[n]$

$$X(\Omega) = \sum_{n=0}^{\infty} 2\left(\frac{3}{4}\right)^n e^{-j\Omega n}$$

$$= 2\sum_{n=0}^{\infty}\left(\frac{3}{4}e^{-j\Omega}\right)^n = 2\frac{1}{1-\frac{3}{4}e^{-j\Omega}}$$

2. a) $x[n] = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$, $N=4$

$$X_k = \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi k}{4} n}$$

$X_0 = 1$

$X_1 = \frac{1}{4}\left(1 + e^{-j\frac{2\pi}{4}} + e^{-j\frac{2\pi}{4} 2} + e^{-j\frac{2\pi}{4} 3}\right)$
$= \frac{1}{4}\left(1 + (-j) - 1 + j\right) = 0$

$X_2 = \frac{1}{4}\left(1 + e^{-j\frac{4\pi}{4}} + e^{-j\frac{8\pi}{4}} + e^{-j\pi}\right)$
$= \frac{1}{4}\left(1 + (-1) + 1 - 1\right) = 0$

$X_3 = \frac{1}{4}\left(1 + e^{-j\frac{3\pi}{2}} + e^{-j3\pi} + e^{-j\frac{9\pi}{2}}\right) = 0$

$$X_k = X(\Omega)\Big|_{\Omega = \frac{2\pi k}{4}}$$

$X(\Omega) = \frac{1}{2} e^{-j3\Omega/2}\left(\cos\left(\frac{3\Omega}{2}\right) + \cos\left(\frac{\Omega}{2}\right)\right)$

$X(0) = 1 = X_0$

$X\left(\frac{2\pi}{4}\right) = 0 = X_1$

$X\left(\frac{2\pi}{4}(2)\right) = 0 = X_2$

$X\left(\frac{\pi}{2}3\right) = 0 = X_3$

2 b) $x[n] = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$

$$X_k = \sum_{n=0}^{2} x[n] e^{-j \frac{2\pi}{3} k n}$$

$X_0 = 0$

$$X_1 = 1 - 2e^{-j \frac{2\pi}{3}} + e^{-j \frac{4\pi}{3}}$$
$$= 1 - 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}j\right) + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}j\right) = \frac{5}{2} + \frac{3\sqrt{3}}{2}j$$

$$X_2 = 1 - 2e^{-j \frac{2\pi}{3} 2} + e^{-j \frac{2\pi}{3} 4}$$
$$= 1 - 2\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = -2 - \frac{3\sqrt{3}}{2}j$$

$$X(\Omega) = 2e^{-j(\Omega \pm \pi)} \, 2\left(1 - \cos(\Omega)\right)$$

$$X_k = X(\Omega)\Big|_{\Omega = \frac{2\pi k}{3}}$$

$$X_0 = X(0) = 0$$

$$X_1 = X\left(\frac{2\pi}{3}\right) = 1.5 + 2.598j$$

$$X_2 = X\left(\frac{4\pi}{3}\right) = 1.5 - 2.598$$

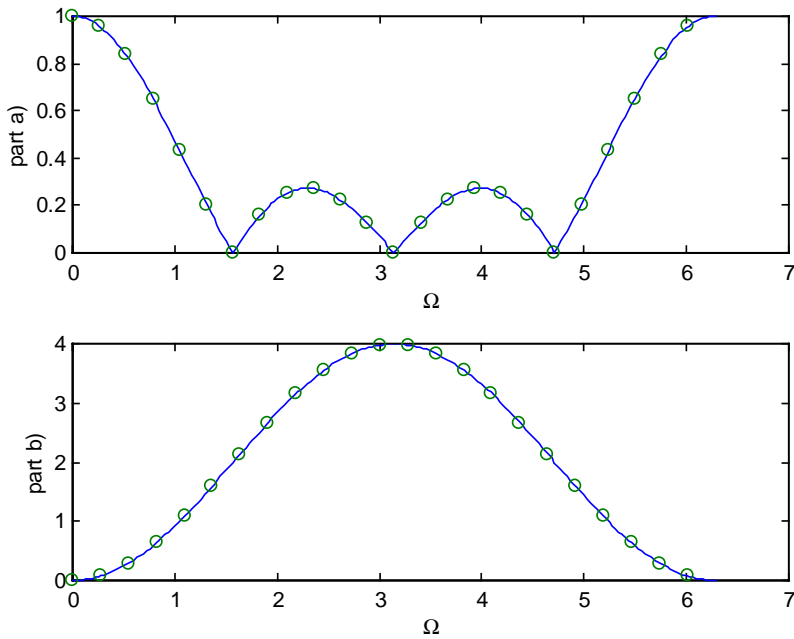3. The MATLAB code used to generate the plots is given below.

```
%a)
x = [.25 .25 .25 .25 zeros(1,20)];
X = dft(x);
Omega = 0:pi/100:pi*2;
Xa = .5*exp(-j*3*Omega/2).*(cos(3*Omega/2)+cos(Omega/2));
N = length(x);
k = 0:N-1;
figure(1)
subplot(211),plot(Omega,abs(Xa),2*k*pi/N,abs(X),'o')
xlabel('\Omega')
ylabel('part a)')
%b)
x = [1 -2 1 zeros(1,20)];
X = dft(x);
Xa = 2*exp(-j*Omega).*(cos(Omega)-1);
N = length(x);
k = 0:N-1;
subplot(212),plot(Omega,abs(Xa),2*k*pi/N,abs(X),'o')
xlabel('\Omega')
ylabel('part b)')

%c)
n = 0:5;
x = 2*(.75).^n;
X = dft(x);
Xa = 2./(1-.75*exp(-j*Omega));
N = length(x);
k = 0:N-1;
figure(2)
subplot(311),plot(Omega,abs(Xa),2*k*pi/N,abs(X),'o')
xlabel('\Omega')
title('part c)')
ylabel('n = 0:5')

n = 0:10;
x = 2*(.75).^n;
X = dft(x);
N = length(x);
k = 0:N-1;
subplot(312),plot(Omega,abs(Xa),2*k*pi/N,abs(X),'o')
xlabel('\Omega')
ylabel('n = 0:10')

n = 0:15;
x = 2*(.75).^n;
X = dft(x);
N = length(x);
k = 0:N-1;
subplot(313),plot(Omega,abs(Xa),2*k*pi/N,abs(X),'o')
xlabel('\Omega')
ylabel('n = 0:15')
```
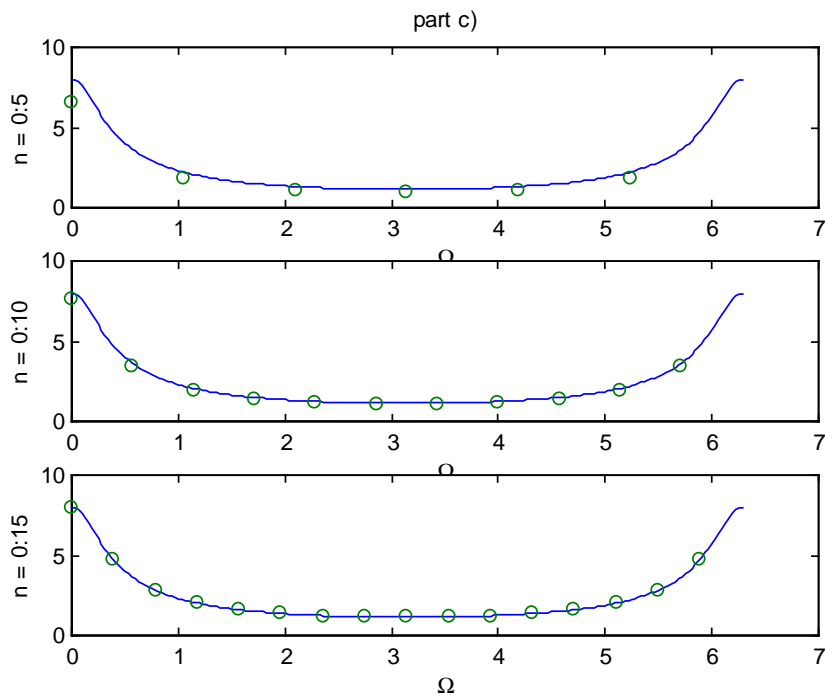
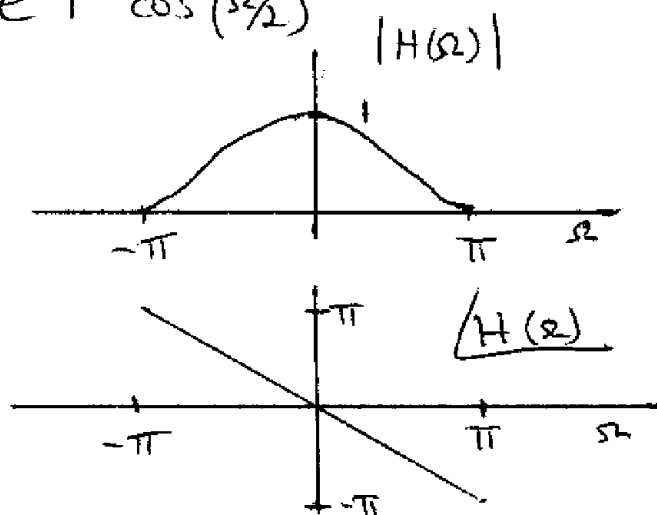The DFT (circles) and DTFT (solid line) for parts a) and b) , exact match since x[n] is finite duration.

The DFT (circles) and the DTFT (solid line) for part c). This is an infinite duration signal that goes to zero. As more points are taken in the sequence (N gets bigger), the more accurate the DFT for approximating the DTFT.

4. $$x[n] = 2 + 2\cos\left(\frac{n\pi}{4}\right) + \cos\left(n\frac{\pi 2}{3} + \frac{\pi}{2}\right)$$

a) $$H(\Omega) = e^{-j\Omega}\cos(\Omega/2)$$

lowpass filter

$$H(0) = 1$$

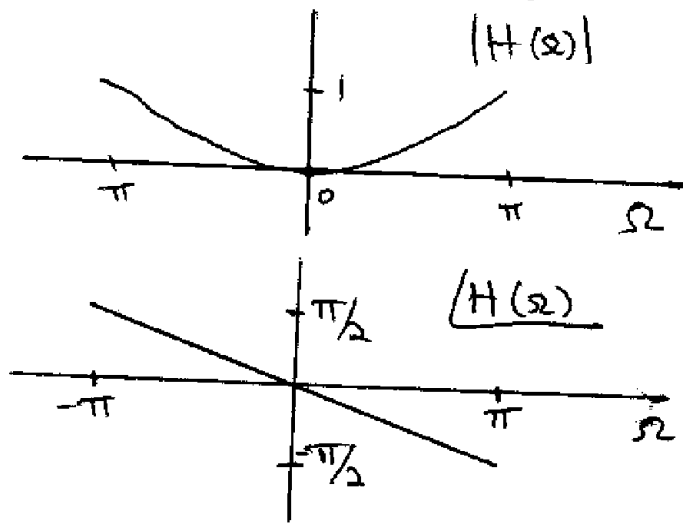$$H\left(\frac{\pi}{4}\right) = e^{-j\pi/4}\cos\left(\frac{\pi}{8}\right) = 0.92\,e^{-j\pi/4}$$

$$H\left(\frac{\pi}{3}\right) = e^{-j2\pi/3}\cos\left(\frac{\pi}{6}\right) = 0.5\,e^{-j2\pi/3}$$

in general:

$$A\cos(\Omega_1 n + \theta) \longrightarrow \boxed{H(\Omega)} \longrightarrow A\,|H(\Omega_1)|\cos\left(\Omega_1 n + \theta + \angle H(\Omega_1)\right)$$

$$y[n] = 2 + 1.84\cos\left(\frac{\pi n}{4} - \frac{\pi}{4}\right) + 0.5\cos\left(\frac{2\pi}{3}n - \frac{\pi}{6}\right)$$

b) $H(\Omega) = e^{-j\Omega/2}\left(1 - \cos(\Omega/2)\right)$

highpass filter

$$x[n] = 2 + 2\cos\left(\frac{\pi}{4}n\right) + \cos\left(\frac{2\pi}{3}n + \frac{\pi}{2}\right)$$

$H(0) = 0$

$$H(\pi/4) = e^{-j\pi/8}\left(1 - \cos(\pi/8)\right) = 0.076\, e^{-j\,\pi/8}$$

$$H\left(\frac{2\pi}{3}\right) = e^{-j\,2\pi/6}\left(1 - \cos\left(\frac{2\pi}{6}\right)\right) = 0.5\, e^{-j\,\pi/3}$$

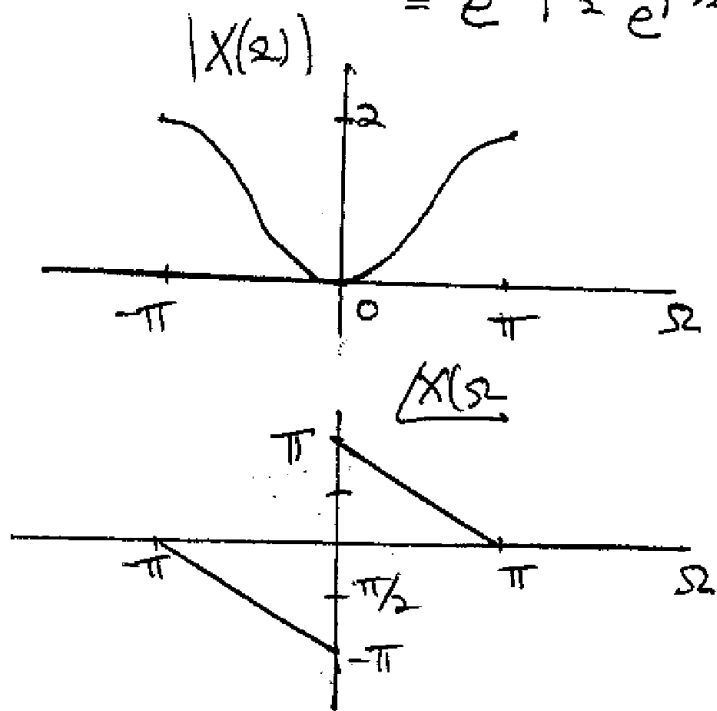$$y[n] = 0.152\cos\left(\frac{\pi}{4}n - \frac{\pi}{8}\right) + 0.5\cos\left(\frac{2\pi}{3}n + \frac{\pi}{6}\right)$$

5    $x[n] = (1 \quad \overset{n=0}{\overset{\swarrow}{-1})}$

DTFT: $X(\Omega) = \sum\limits_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

$= e^0 - e^{-j\Omega} = e^{-j\Omega/2}(e^{j\Omega/2} - e^{-j\Omega/2})$

$= e^{-j\Omega/2} \, 2j \sin\left(\frac{\Omega}{2}\right)$

$= e^{-j\frac{\Omega}{2}} e^{\pm j\pi/2} \, 2 \sin\left(\frac{\Omega}{2}\right)$
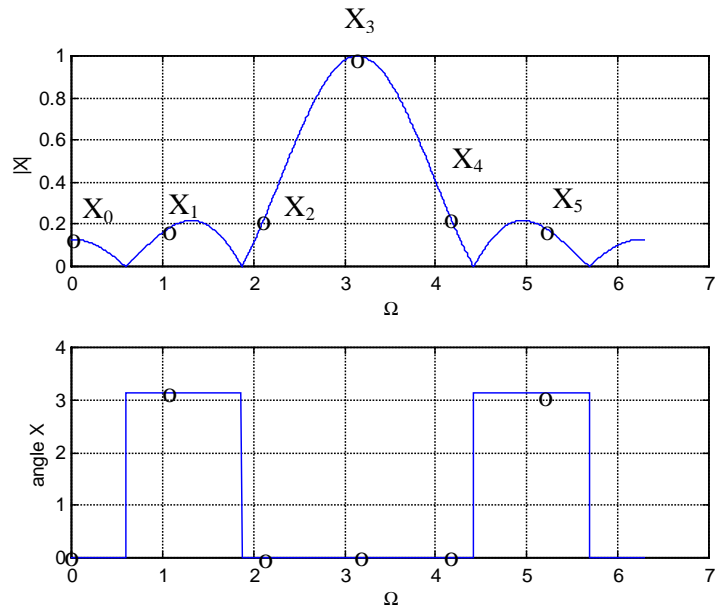

$|X(\Omega)|$

$\angle X(\Omega)$

DFT: $X_k = \sum\limits_{n=0}^{1} x[n] e^{-j\frac{2\pi k}{2} n}$

$= 1 - e^{-j\pi k}$

$X_0 = 1 - 1 = 0$

$X_1 = 1 - e^{-j\pi} = 2$

6. The DFT is a discretized version of the DTFT at the points $\Omega = \dfrac{2\pi k}{N}$ for $k = 0,...,N-1$. In this case,

$N = 6$, so the points are located at $\Omega = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \pi, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$
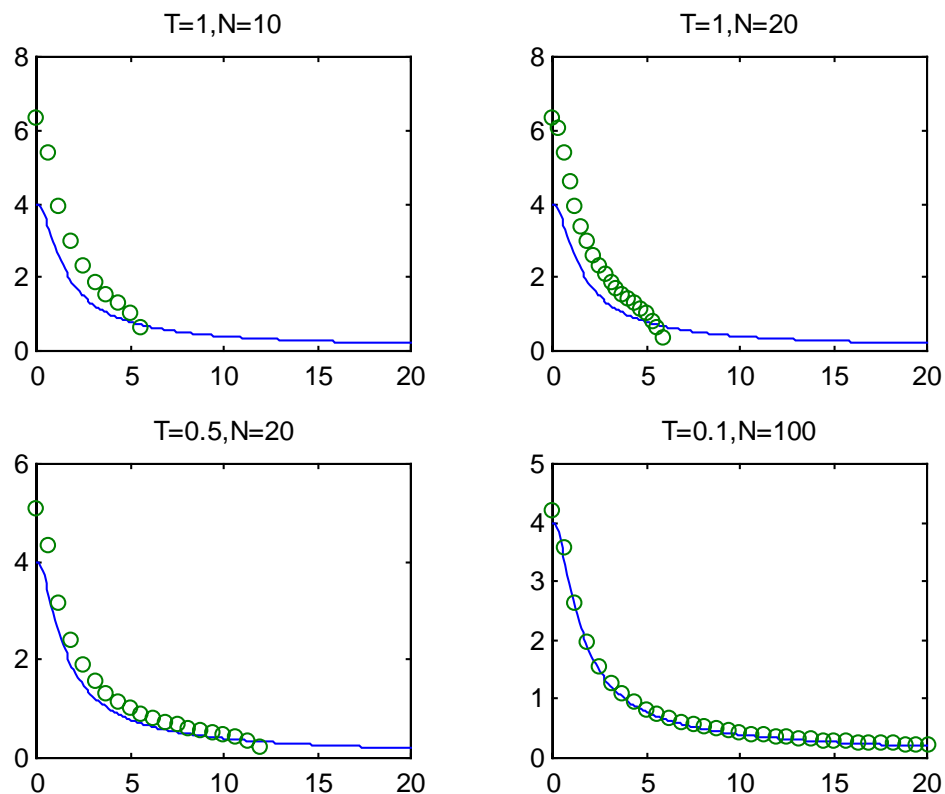
7. The plot gets more accurate as T gets smaller. The resolution gets smaller as NT gets bigger. Note that in a signal that is not time-limited but decays, sample for long enough time so that the truncated part of x(t) is negligible.

```
% actual X(w)
w = 0:.1:20;
Xa = 4./(j*w+1);

T = 1; N = 10;
t = 0:T:T*(N-1);
x = 4*exp(-t);
[Xi,wi] = contfft(x,T);
T = 1; N = 20;
t = 0:T:T*(N-1);
x = 4*exp(-t);
[Xii,wii] = contfft(x,T);
T = 0.5; N = 20;
t = 0:T:T*(N-1);
x = 4*exp(-t);
[Xiii,wiii] = contfft(x,T);
T = 0.1; N = 100;
t = 0:T:T*(N-1);
x = 4*exp(-t);
[Xiv,wiv] = contfft(x,T);
subplot(221),plot(w,abs(Xa),wi,abs(Xi),'o')
title('T=1,N=10')
subplot(222),plot(w,abs(Xa),wii,abs(Xii),'o')
title('T=1,N=20')
subplot(223),plot(w,abs(Xa),wiii,abs(Xiii),'o')
title('T=0.5,N=20')
subplot(224),plot(w,abs(Xa),wiv,abs(Xiv),'o')
axis([0 20 0 5])
title('T=0.1,N=100')
```

8. $x(t) = 2e^{-2t} u(t)$

     $X(\omega) = \dfrac{2}{j\omega + 2}$

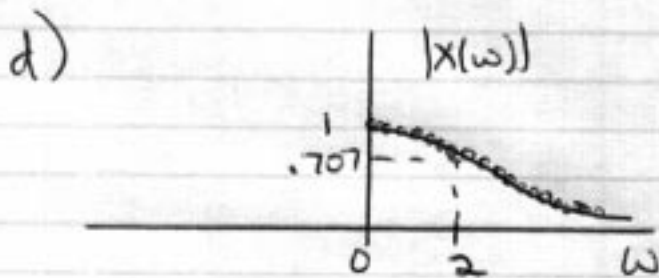a) $|X(\omega)| = \dfrac{2}{\sqrt{\omega^2 + 2^2}} \leq .02 \;|X(0)| = 0.02$

     $\dfrac{1}{}$ $100 \leq \sqrt{\omega^2 + 2^2}$

         $\sqrt{100^2 - 4} \leq W_B \;\Rightarrow\; W_B \approx 100$

b) $W_s > 2W_B = 200 \;\Rightarrow\; T = \dfrac{2\pi}{200} = \dfrac{\pi}{100}$

c) $\Gamma = \dfrac{2\pi}{NT} \leq .2$

     $\dfrac{2\pi}{.2\left(\pi/100\right)} = N = 1000 \text{ pts.}$

d)

should be very
close
both plots are given
below → almost identical

|X| exact and approximated