Corruption, Justice, and Democracy in Compressive Sensing

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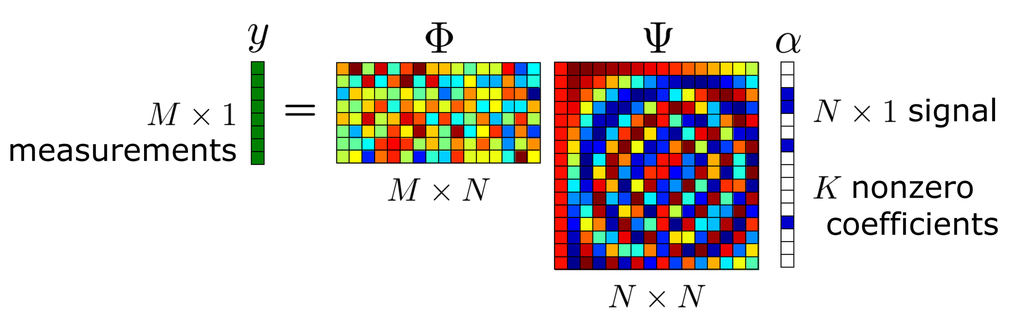
Compressive Sensing

Background

Directly acquire a reduced set of low-dimensional ***compressive measurements***



Nonlinear recovery via optimization-based, iterative, or greedy algorithms

Basis Pursuit (BP)

$$\widehat{\alpha} = \arg\min_{\alpha} \|\alpha\|_1 \quad \text{s.t.} \quad y = \Phi\Psi\alpha$$

Basis Pursuit De-Noising (BPDN)

$$\widehat{\alpha} = \arg\min_{\alpha} \|\alpha\|_1 \quad \text{s.t.} \quad \|y - \Phi\Psi\alpha\|_2 \le \epsilon$$

The ***restricted isometry property*** (RIP) ensures that Φ captures the information in the signal

$$(1-\delta)\|\alpha\|_2^2 \le \|\Phi\Psi\alpha\|_2^2 \le (1+\delta)\|\alpha\|_2^2 \qquad \forall \alpha \quad \|\alpha\|_0 \le K$$

Theorem [Candès]: Suppose that $y = \Phi\Psi\alpha + e$ and $\|\alpha\|_0 \le K$. If Φ satisfies the RIP of order $2K$ with $\delta < \sqrt{2} - 1$, then the BPDN solution satisfies

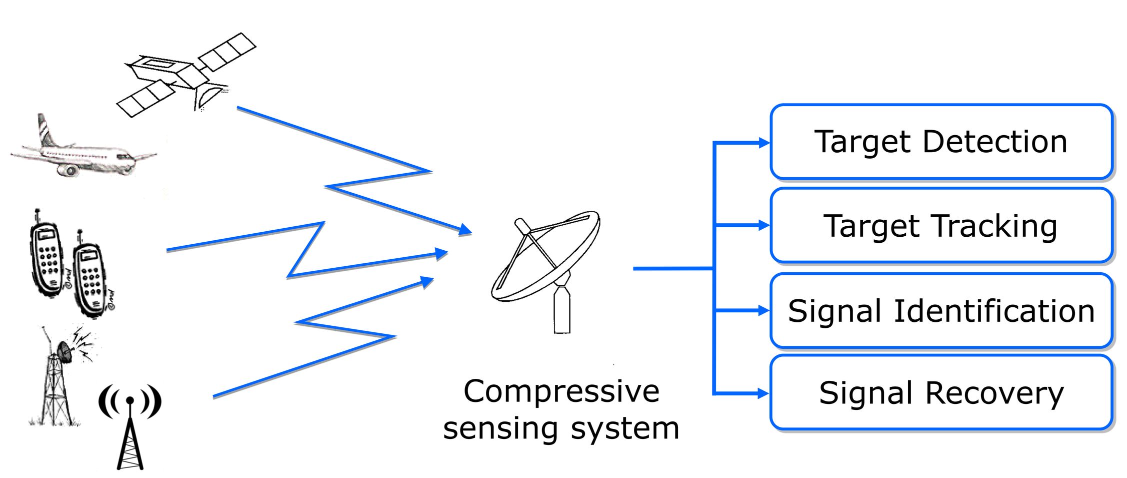
$$\|\widehat{\alpha} - \alpha\|_2 \le C_0 \|e\|_2.$$

SubGaussian Φ satisfy the RIP if $M = O(K \log(N/K))$.

Does randomness provide any other benefits?

Compressive Signal Processing

Random measurements are ***information scalable***



In such scenarios, measurements are often corrupted by ***interference*** and ***structured noise***

$$y = \Phi x_S + \Phi x_I \qquad\qquad y = \Phi x_S + \Omega e$$

Seek to remove contribution of Φx_I or Ωe to y before reconstructing x_S.

Corruption

Interference Cancellation

Assume $x_S \in \mathcal{X}_S$ and $x_I \in \mathcal{X}_I$, where $\langle x_I, x_S \rangle = 0$ for all $x_S \in \mathcal{X}_S$, $x_I \in \mathcal{X}_I$. Also assume $\Psi = I$.

Design $M \times M$ matrix P such that

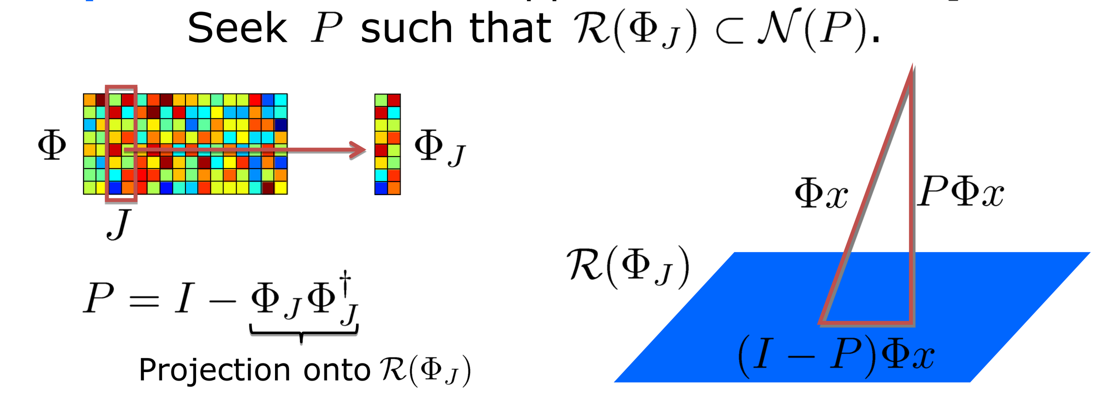
$$\|P(\Phi x_I)\|_2 \approx 0 \quad \text{and} \quad \|P(\Phi x_S)\|_2 \approx \|\Phi x_S\|_2$$

Note: Not always possible

Depends on structure of $\mathcal{X}_S$ and $\mathcal{X}_I$

Subspace Cancellation

Example: x_I has known support set J of size K_I
Seek P such that $\mathcal{R}(\Phi_J) \subset \mathcal{N}(P)$.



Observe that $Py = P\Phi x_S + \cancel{P\Phi x_I}$
$= P\Phi x_S$

Theorem: If Φ satisfies the RIP of order $2K_S + K_I$, then $P\Phi$ satisfies

$$\left(1 - \frac{\delta}{1-\delta}\right)\|x\|_2^2 \le \|P\Phi x\|_2^2 \le (1+\delta)\|x\|_2^2$$

for all x such that $\|x\|_0 \le 2K_S$ and $\mathrm{supp}(x) \cap J = \emptyset$.

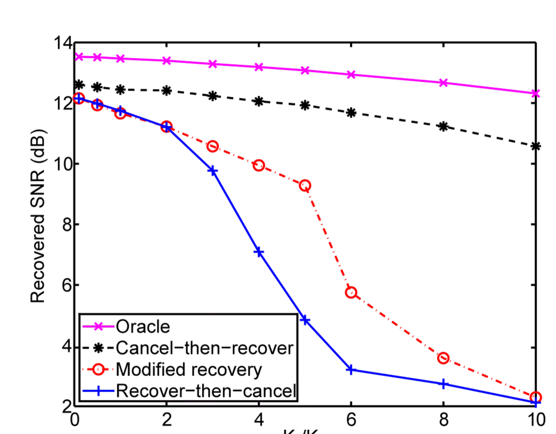
Proof exploits two facts:

$$\|\Phi x\|_2^2 = \|P\Phi x\|_2^2 + \|(I-P)\Phi x\|_2^2$$

$$\frac{\|(I-P)\Phi x\|_2}{\|\Phi x\|_2} = \frac{\langle (I-P)\Phi x, \Phi x\rangle}{\|(I-P)\Phi x\|_2 \|\Phi x\|_2} \le \frac{\delta}{1+\delta}$$

Implications

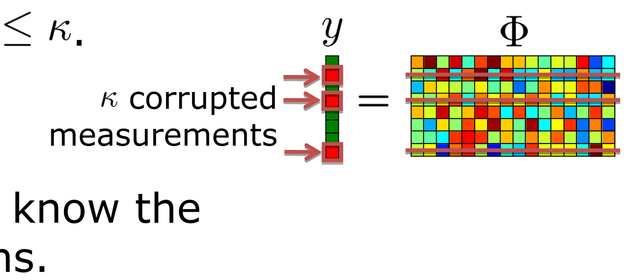
- Allows for **"cancel-then-recover"** approach to signal recovery in interference
- Useful tool in analysis of greedy algorithms such as **Orthogonal Matching Pursuit** (OMP) and **Regularized OMP** (ROMP)



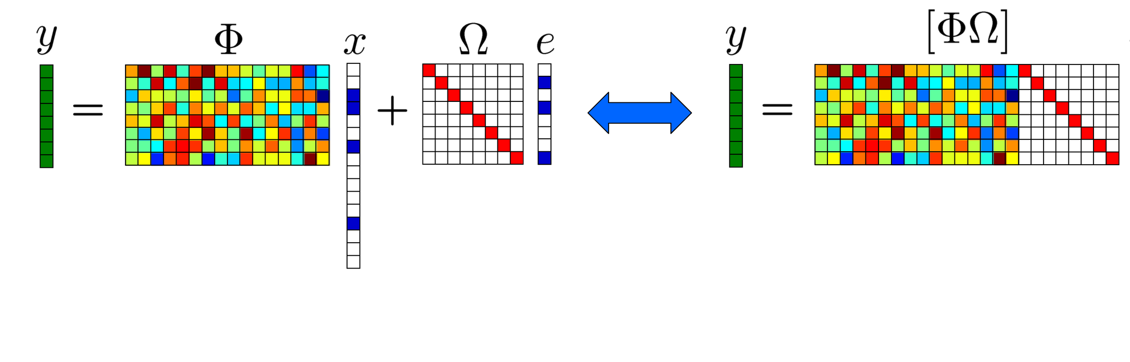
Justice

Corrupted Measurements

Suppose $y = \Phi x + \Omega e$
where $\Omega = I$ and $\|e\|_0 \le \kappa$.



In general, we do not know the locations of corruptions.



"Justice Pursuit" (JP)

$$\widehat{u} = \arg\min_{u} \|u\|_1 \quad \text{s.t.} \quad y = [\Phi\Omega]\, u$$

To analyze Justice Pursuit, we must study the properties of the matrix $[\Phi\Omega]$.

Theorem: If Φ is a subGaussian matrix with

$$M = O\left((K+\kappa)\log\left(\frac{N+M}{K+\kappa}\right)\right)$$

then $[\Phi\Omega]$ satisfies the RIP of order $(K+\kappa)$ with probability at least $1 - 3e^{-CM}$.

Proof follows from

$$\|[\Phi\Omega]\, u\|_2^2 = \|\Phi x\|_2^2 + e^T\Omega^T\Phi x + \|e\|_2^2$$

and the facts that with high probability

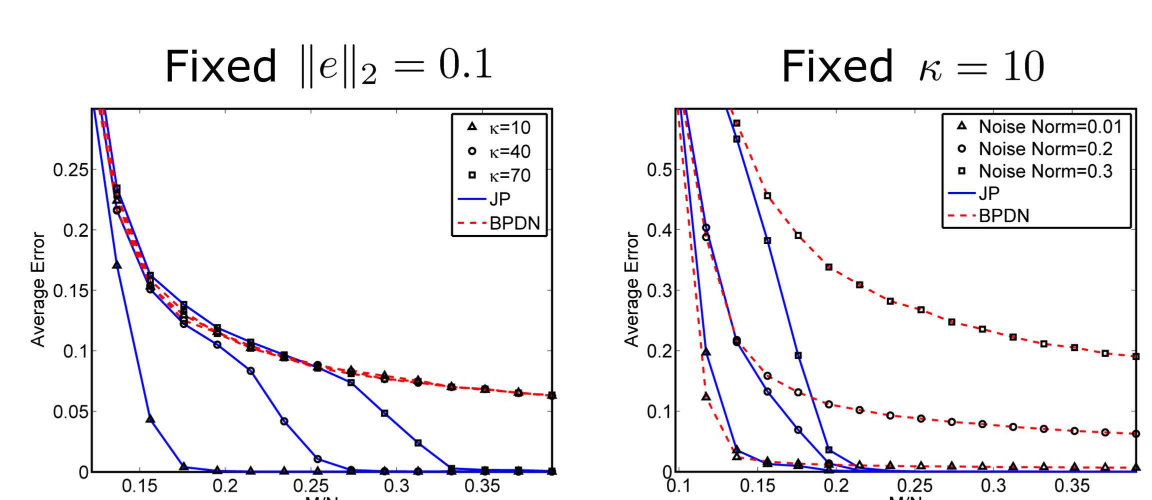
$$-\delta\|e\|_2\|x\|_2 \le e^T\Omega^T\Phi x \le \delta\|e\|_2\|x\|_2$$

$$(1-\delta)\|x\|_2^2 \le \|\Phi x\|_2^2 \le (1+\delta)\|x\|_2^2$$

Experiments

Compare JP with BPDN ($N = 1024$, $K = 10$)

If M is sufficiently large, JP achieves ***exact*** recovery



Democracy

Corruption meets Justice

The key results of subspace cancellation and justice combine to provide a simple proof that random matrices are ***democratic***.

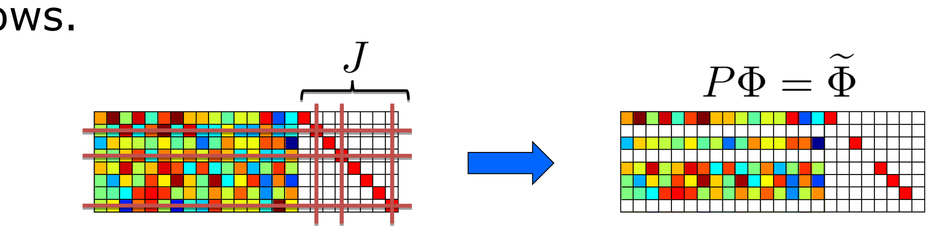
A matrix is democratic if we can remove D arbitrary (adversarially selected) rows and retain the RIP.

If

$$M = O\left((K+D)\log\left(\frac{N+M}{K+D}\right)\right)$$

then $[\Phi I]$ satisfies the RIP of order $(K+D)$.

Construct P to cancel interference from columns indexed by J, where J corresponds to a set of D rows.



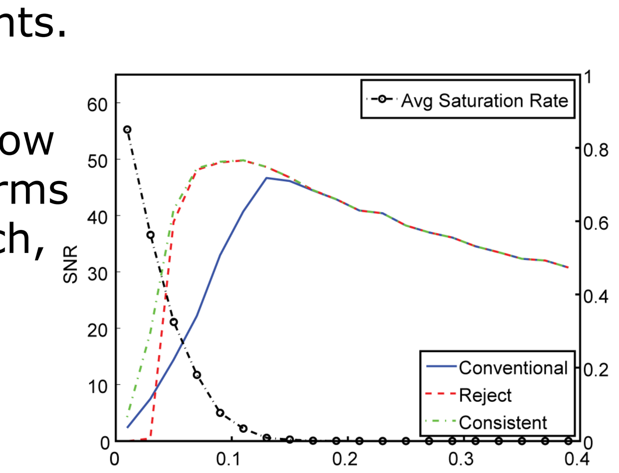
Since $\widetilde{\Phi}$ will satisfy the RIP for any possible choice of J, this establishes that Φ is democratic.

Democracy in Action

When measurements are quantized using a finite-range quantizer, some will ***saturate***.

Democracy justifies a strategy of simply ***rejecting*** saturated measurements.

In fact, simulations show this method out-performs the traditional approach, achieving optimal performance at non-zero saturation rates.



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