1-Bit Matrix Completion

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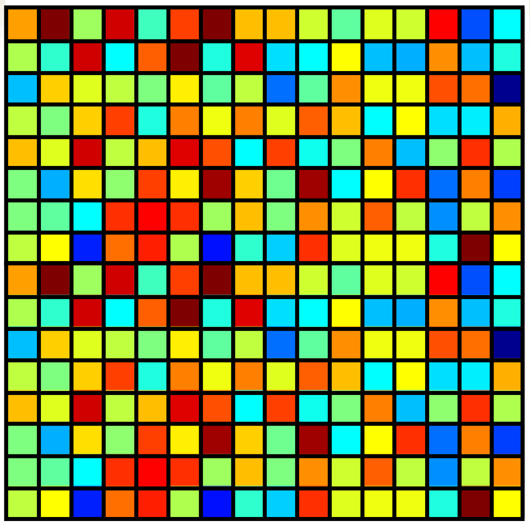
Matrix Completion



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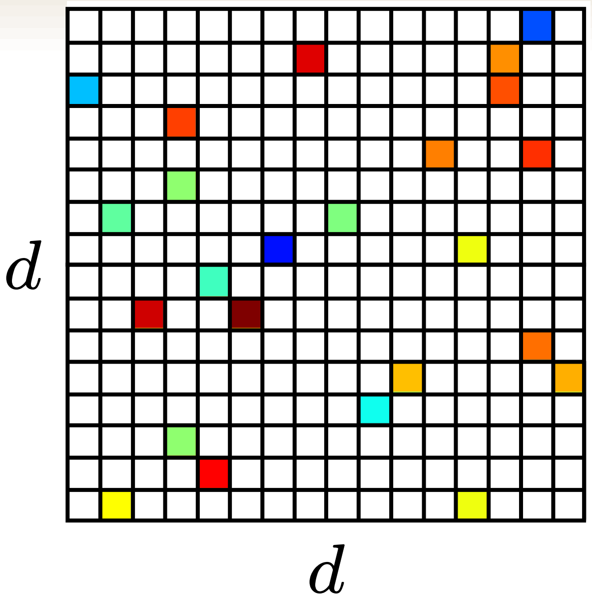
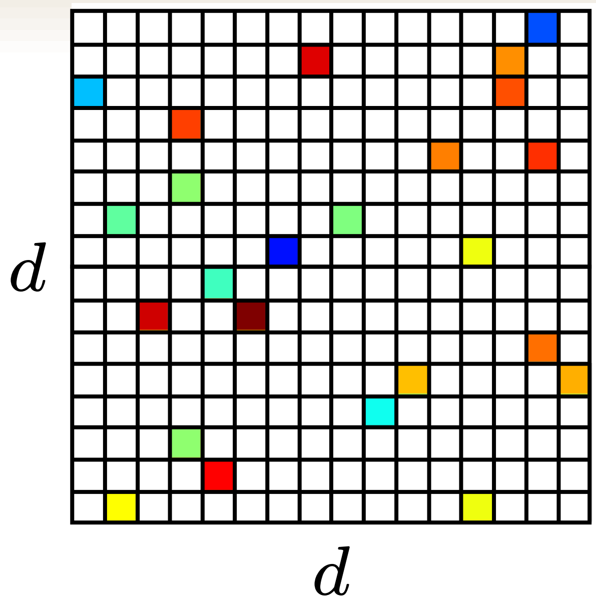
Matrix Completion

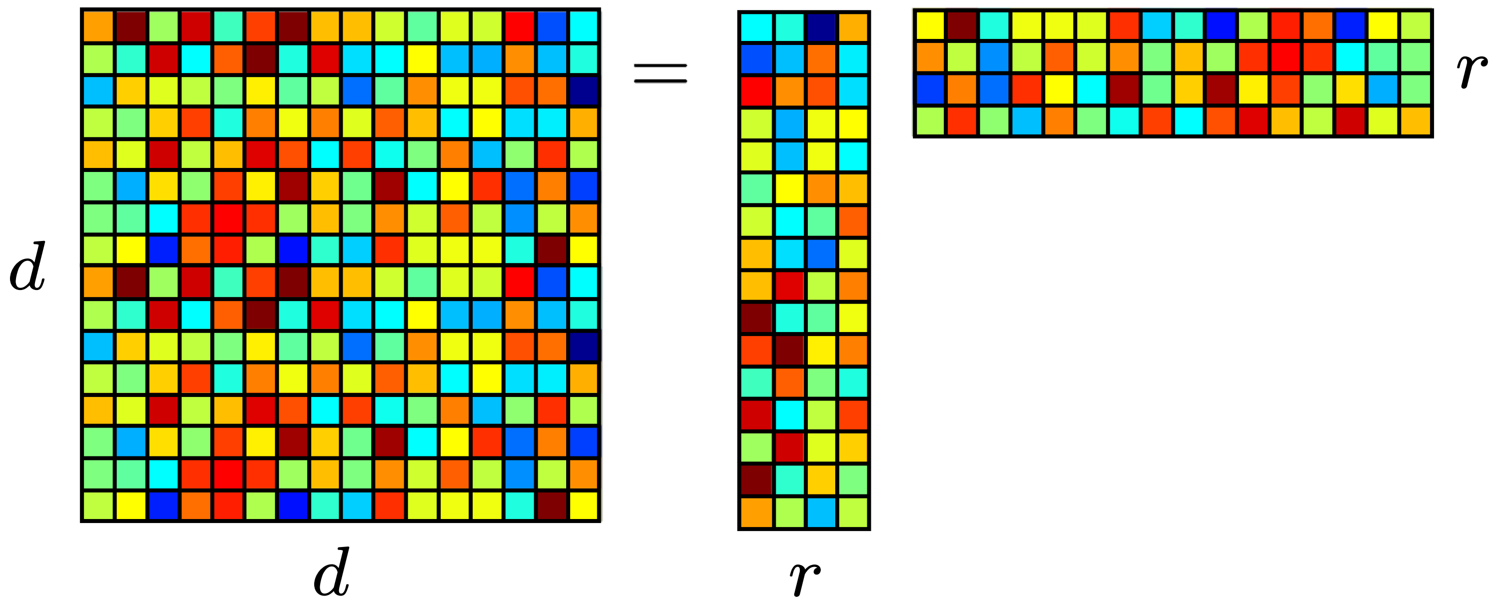


Matrix Completion



- When is it possible to recover the original matrix?
- How can we do this efficiently?
- How many samples will we need?

Low-Rank Matrices



Singular value decomposition:

$$M = U\Sigma V^* \quad \Rightarrow \quad \approx dr \ll d^2$$

degrees of freedom

Low-Rank Matrix Recovery

Given:

- a $d \times d$ matrix M of rank r
- samples of M on the set : $Y = M$

How can we recover M?

$$\widehat{M} = \arg\inf_{X : X \quad = Y} \mathrm{rank}(X)$$

Nuclear Norm Minimization

Convex relaxation!

Replace $\operatorname{rank}(X)$ with $\|X\|_* = \sum_{j=1}^{d} |\sigma_j|$

$$\widehat{M} = \operatorname*{arg\,inf}_{X : X \quad = Y} \|X\|_*$$

If $| \quad | = O(r\,d \log d)$, this procedure can recover M !

Applications

- Collaborative Filtering (aka the “Netflix Problem”)
- Recovery of incomplete survey data
- Analysis of voting data
- Sensor localization
- Quantum state tomography
- …

Matrix Completion in Practice

- Noise

$$Y = (M + Z)$$

Matrix Completion in Practice

- Noise

$$Y = (M + Z)$$

- ***Quantization***
  - Netflix: Ratings are integers between 1 and 5
  - Survey responses: True/False, Yes/No, Agree/Disagree
  - Voting data: Yea/Nay
  - Quantum state tomography: Binary outcomes

Matrix Completion in Practice

- Noise

$$Y = (M + Z)$$

- ***Quantization***
  - Netflix: Ratings are integers between 1 and 5
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Extreme quantization *destroys low-rank structure* and makes recovery *highly ill-posed*

1-Bit Matrix Completion

Extreme case

$$Y = \mathrm{sign}(M \quad)$$

Claim: Recovering M from Y is impossible!

$$M = \begin{bmatrix} \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda \end{bmatrix}$$

No matter how many samples we obtain, all we can learn is whether $\lambda > 0$ or $\lambda < 0$

Is There Any Hope?

If we consider a noisy version of the problem, recovery becomes feasible!

$$Y = \mathrm{sign}(M \quad + Z \quad)$$

$$M + Z = \begin{bmatrix} \lambda + Z_{1,1} & \lambda + Z_{1,2} & \lambda + Z_{1,3} & \lambda + Z_{1,4} \\ \lambda + Z_{2,1} & \lambda + Z_{2,2} & \lambda + Z_{2,3} & \lambda + Z_{2,4} \\ \lambda + Z_{3,1} & \lambda + Z_{3,2} & \lambda + Z_{3,3} & \lambda + Z_{3,4} \\ \lambda + Z_{4,1} & \lambda + Z_{4,2} & \lambda + Z_{4,3} & \lambda + Z_{4,4} \end{bmatrix}$$

Fraction of positive/negative observations tells us something about λ

Example of the power of *dithering*

Observation Model

For $(i, j) \in$ we observe

$$Y_{i,j} = \begin{cases} +1 & \text{with probability } f(M_{i,j}) \\ -1 & \text{with probability } 1 - f(M_{i,j}) \end{cases}$$

If f behaves like a CDF, then this is equivalent to

$$Y_{i,j} = \operatorname{sign}(M_{i,j} + Z_{i,j})$$

where $Z_{i,j}$ is drawn according to a suitable distribution

We will assume that is drawn uniformly at random and that $\|M\|_\infty \leq \alpha$

Examples

- Logistic regression / Logistic noise

$$f(x) = \frac{e^x}{1 + e^x}$$

$$Z_{i,j} \sim \text{logistic distribution}$$

- Probit regression / Gaussian noise

$$f(x) = \Phi(x/\sigma)$$

$$Z_{i,j} \sim \mathcal{N}(0, \sigma^2)$$

Maximum Likelihood Estimation

Log-likelihood function:

$$F(X) = \sum_{(i,j)\in \quad _{+}} \log(f(X_{i,j})) + \sum_{(i,j)\in \quad _{-}} \log(1 - f(X_{i,j}))$$

$$\begin{aligned} \widehat{M} &= \arg\max_{X} F(X) \\ &\qquad \text{s.t.} \ \ \operatorname{rank}(X) \le r \\ &\qquad\qquad \|X\|_\infty \le \alpha \end{aligned}$$

Maximum Likelihood Estimation

Log-likelihood function:

$$F(X) = \sum_{(i,j)\in \quad +} \log(f(X_{i,j})) + \sum_{(i,j)\in \quad -} \log(1 - f(X_{i,j}))$$

$$\begin{aligned} \widehat{M} = \arg\max_{X} F(X) \\ \text{s.t. } \frac{1}{d\alpha}\|X\|_* \leq \sqrt{r} \\ \|X\|_\infty \leq \alpha \end{aligned}$$

Recovery of the Matrix

***Theorem** (Upper bound achieved by convex ML estimator)*

Assume that $\frac{1}{d\alpha}\|M\|_* \leq \sqrt{r}$ and $\|M\|_\infty \leq \alpha$. If is chosen at random with $\mathbb{E}|\quad| = m > d \log d$, then with high probability

$$\frac{1}{d^2}\|\widehat{M} - M\|_F^2 \leq C\alpha L_\alpha \beta_\alpha \sqrt{\frac{rd}{m}}$$

where

$$L_\alpha := \sup_{|x| \leq \alpha} \frac{|f'(x)|}{f(x)(1 - f(x))} \qquad \beta_\alpha := \sup_{|x| \leq \alpha} \frac{f(x)(1 - f(x))}{(f'(x))^2}$$

Is this bound tight?

Recovery of the Matrix

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***Theorem** (Lower bound on any estimator)*

For any recovery algorithm $\widehat{M}$ there exist M satisfying the assumptions above such that for any set   with $|\quad| = m$, we have (under mild technical assumptions) that

$$\mathbb{E}\left[\frac{1}{d^2}\|\widehat{M} - M\|_F^2\right] \geq c\alpha\sqrt{\beta_{\frac{3}{4}\alpha}}\sqrt{\frac{rd}{m}}$$

Conclusions

- In the stochastic setting, 1-bit matrix completion is possible via a simple convex algorithm

- Proof techniques
  - Upper bounds: random matrix theory
  - Lower bounds: Fano's inequality, packing sets of low-rank matrices

- More noise helps, but only up to a point

Thank You!