# Compressive Sensing in Noise and the Role of Adaptivity

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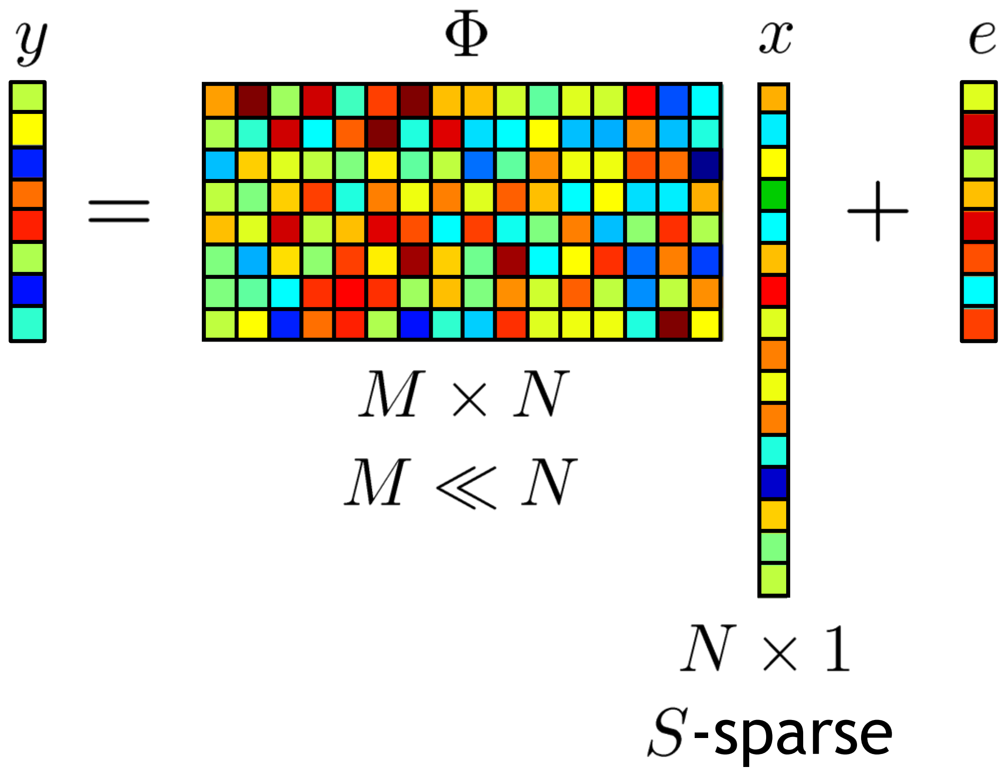
# Compressive Sensing in Noise




When (and how well) can we estimate $x$ from the measurements $y$?

# Nonadaptive Compressive Sensing

# Stable Signal Recovery

Given $y = \Phi x + e$ ,
find $x$

Typical (worst-case) guarantee: If $\Phi$ satisfies the RIP

$$\|\widehat{x} - x\|_2^2 \leq C\|e\|_2^2$$

Even if $\Lambda = \mathrm{supp}(x)$ is provided by an oracle, the error can still be as large as $\|\widehat{x} - x\|_2^2 = \|e\|_2^2/(1-\delta)$ .

# Stable Signal Recovery: Part II

Suppose now that $\Phi$ satisfies

$$A(1-\delta)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq A(1+\delta)\|x\|_2^2 \qquad \|x\|_0 \leq 2S$$

In this case our guarantee becomes

$$\|\widehat{x} - x\|_2^2 \leq \frac{C}{A}\|e\|_2^2$$

Unit-norm rows $\Rightarrow$ $\|\widehat{x} - x\|_2^2 \leq C\frac{N}{M}\|e\|_2^2$

# Expected Performance

- Worst-case bounds can be pessimistic
- What about the *average* error?
  - assume $e$ is white noise with variance $\sigma^2$

$$\mathbb{E}\left(\|e\|_2^2\right) = M\sigma^2$$

  - for oracle-assisted estimator

$$\mathbb{E}\left(\|\widehat{x} - x\|_2^2\right) \leq \frac{S\sigma^2}{A(1-\delta)}$$

  - if $e$ is Gaussian, then for $\ell_1$ -minimization

$$\mathbb{E}\left(\|\widehat{x} - x\|_2^2\right) \leq \frac{C'}{A} S\sigma^2 \log N$$

# White Signal Noise

What if our signal $x$ is contaminated with noise?

$$y = \Phi(x + n) = \Phi x + \Phi n$$

Suppose $\Phi$ has orthogonal rows with norm equal to $\sqrt{B}$.
If $n$ is white noise with variance $\sigma^2$, then $\Phi n$ is white noise with variance $B\sigma^2$.

$$\mathbb{E}\left[\|\widehat{x} - x\|_2^2\right] \leq C' \frac{B}{A} S \sigma^2 \log N$$

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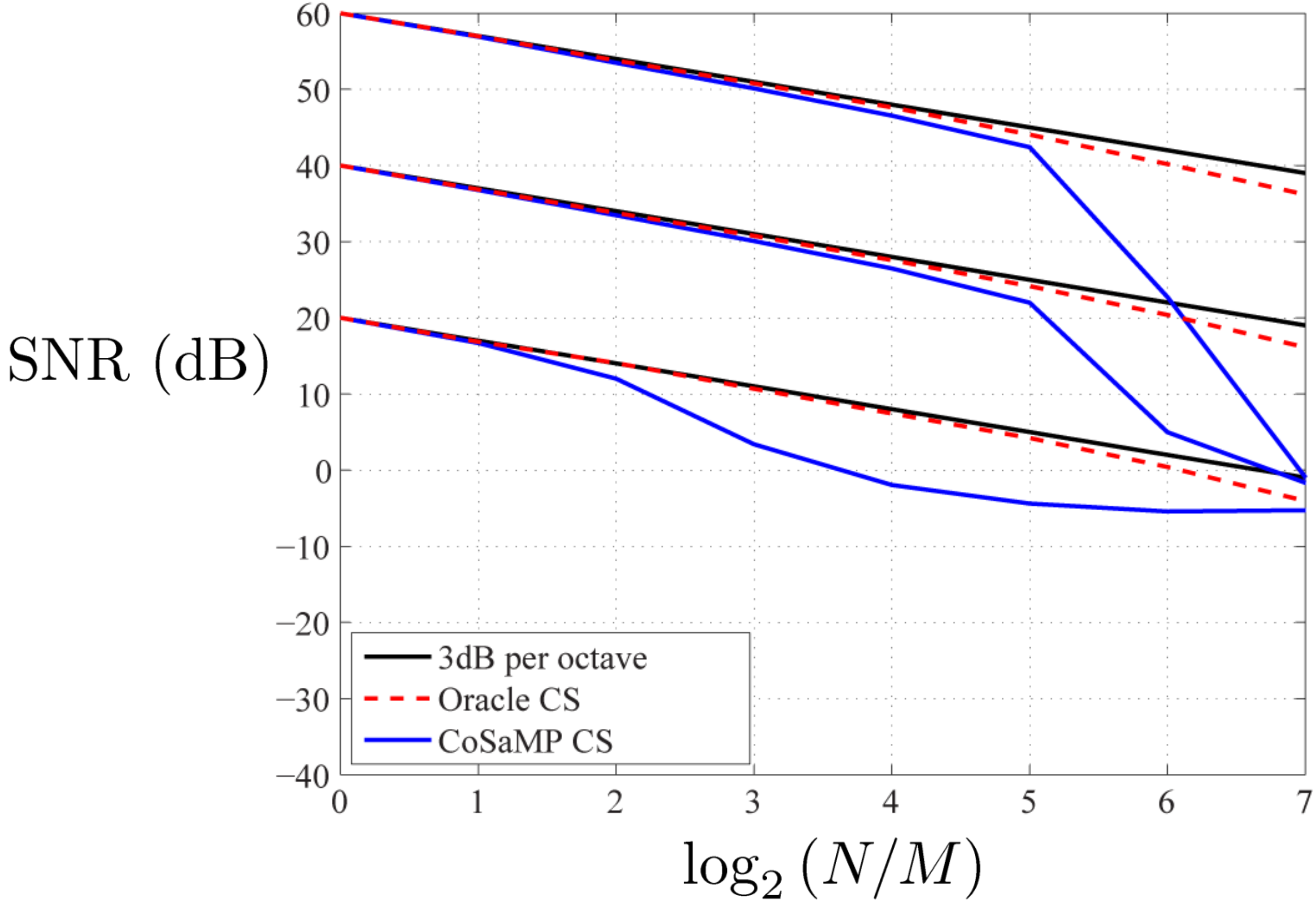
$$\mathbb{E}\left[\|\widehat{x}-x\|_2^2\right] \leq C' \frac{N}{M} S \sigma^2 \log N$$

$$\mathrm{SNR} = 10 \log_{10}\left(\frac{\|x\|_2^2}{\|\widehat{x}-x\|_2^2}\right)$$

→ 3dB loss per octave of subsampling

# Noise Folding




[Davenport, Laska, Treichler, Baraniuk - 2011]

# Room For Improvement?

There exists matrices $\Phi$ (with unit-norm rows) such that for *any* (sparse) $x$ we have

$$\mathbb{E}\,\|\widehat{x} - x\|_2^2 \leq C\frac{N}{M}S\sigma^2 \log N.$$

$$y_i = \langle \phi_i, x \rangle + e_i$$

↑

$\phi_i$ and $x$ are almost orthogonal

- We are using most of our “sensing power” to sense entries that aren’t even there!
- Tremendous loss in signal-to-noise ratio (SNR)
- It’s hard to imagine any way to avoid this…

# Can We Do Better?

Via a better choice of $\Phi$? Via a better recovery algorithm?

If $y = \Phi x + e$ with $e \sim \mathcal{N}(0, \sigma^2 I)$, then there exists an $x$ such that for *any* $\widehat{x}$ and *any* $\Phi$

$$\mathbb{E}\left[\|\widehat{x}(\Phi x + e) - x\|_2^2\right] \geq C \frac{N}{\|\Phi\|_F^2} S\sigma^2 \log(N/S).$$

If $y = \Phi(x + n)$ with $n \sim \mathcal{N}(0, \sigma^2 I)$, then there exists an $x$ such that for *any* $\widehat{x}$ and *any* $\Phi$

$$\mathbb{E}\left[\|\widehat{x}(\Phi(x + n)) - x\|_2^2\right] \geq C \frac{N}{M} S\sigma^2 \log(N/S).$$

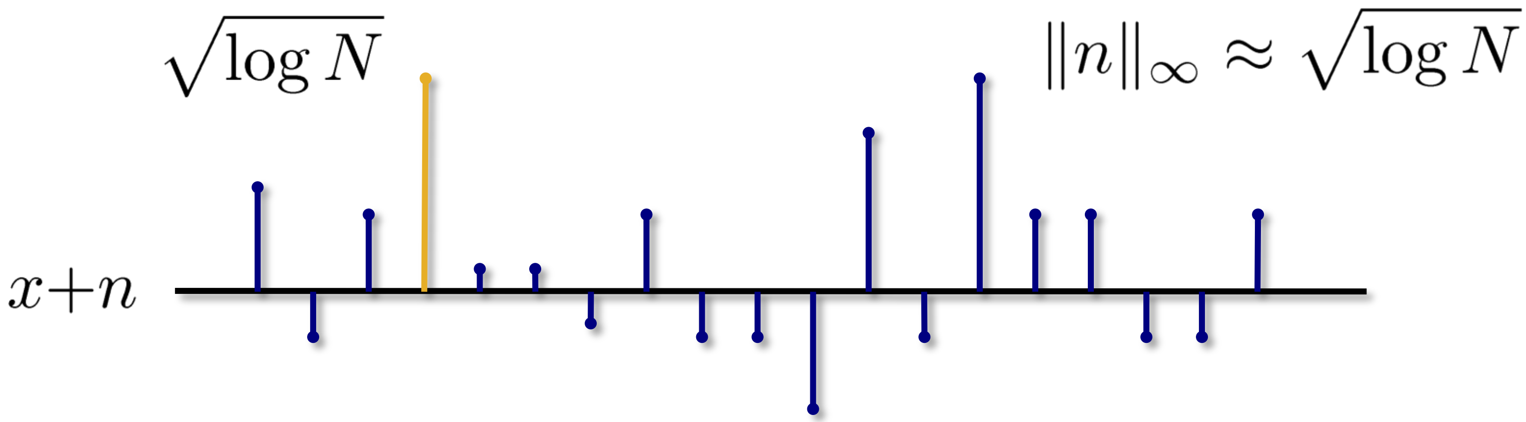
$$\Phi = U\Sigma V^* \quad y' = \Sigma^{-1}U^* y = V^* x + V^* n \quad \|V^*\|_F^2 = M$$

[Candès and Davenport - 2011]

# Intuition

Suppose that $y = x + n$ with $n \sim \mathcal{N}(0, I)$ and that $S = 1$

$$\mathbb{E}\,\|\widehat{x}(y) - x\|_2^2 \geq C' \log N$$



# Proof Recipe

*Ingredients* (Makes $\sigma^2 = 1$ servings)

- Lemma 1: There exists a set $\mathcal{X}$ of $S$-sparse vectors such that
  - $|\mathcal{X}| = (N/S)^{S/4}$
  - $\|x_i - x_j\|_2 \geq \frac{1}{2}$ for all $x_i, x_j \in \mathcal{X}$
  - $\| \frac{1}{|\mathcal{X}|} \sum_i x_i x_i^* - \frac{1}{N} I \| \leq \frac{\beta}{N}$ for some $\beta > 0$
- Lemma 2: Define $R^*_{\mathrm{mm}}(\Phi) = \inf_{\widehat{x}} \sup_{\|x\|_0 \leq S} \mathbb{E}\left[\|\widehat{x}(\Phi x + e) - x\|_2^2\right]$.

  Suppose $\mathcal{X}$ is a set of $S$-sparse vectors such that $\|x_i - x_j\|_2^2 \geq 8 N R^*_{\mathrm{mm}}(\Phi)$ for all $x_i, x_j \in \mathcal{X}$.
  Then $\frac{1}{2} \log |\mathcal{X}| - 1 \leq \frac{1}{2|\mathcal{X}|^2} \sum_{i,j} \|\Phi x_i - \Phi x_j\|_2^2$.

*Instructions*

Combine ingredients and add a dash of linear algebra.

# The Details

$$\mu = \frac{1}{|\mathcal{X}|}\sum_i x_i \qquad Q = \frac{1}{|\mathcal{X}|}\sum_i x_i x_i^*$$

$$\begin{aligned}
\frac{S}{4}\log(N/S) - 2 &\le \frac{1}{|\mathcal{X}|^2}\sum_{i,j}\|\Phi x_i - \Phi x_j\|_2^2 \\
&= \mathrm{Tr}\left(\Phi^*\Phi\left(\frac{1}{|\mathcal{X}|^2}\sum_{i,j}(x_i - x_j)(x_i - x_j)^*\right)\right) \\
&= \mathrm{Tr}\left(\Phi^*\Phi\left(2(Q - \mu\mu^*)\right)\right) \\
&\le 2\mathrm{Tr}\left(\Phi^*\Phi Q\right) \\
&\le 2\mathrm{Tr}\left(\Phi^*\Phi\right)\|Q\| \\
&\le 2\|\Phi\|_F^2 \cdot 16 R_{\mathrm{mm}}^*(\Phi)(1+\beta)
\end{aligned}$$

$$\Rightarrow \quad R_{\mathrm{mm}}^*(\Phi) \ge \frac{S\log(N/S)}{128(1+\beta)\|\Phi\|_F^2}$$

# Lemma 1

Lemma 1: There exists a set $\mathcal{X}$ of $S$-sparse points such that

- $|\mathcal{X}| = (N/S)^{S/4}$
- $\|x_i - x_j\|_2 \geq \frac{1}{2}$ for all $x_i, x_j \in \mathcal{X}$
- $\|\frac{1}{|\mathcal{X}|} \sum_i x_i x_i^* - \frac{1}{N} I\| \leq \frac{\beta}{N}$ for some $\beta > 0$

## *Strategy*

Construct $\mathcal{X}$ by sampling (with replacement) from

$$\mathcal{U} = \left\{ x \in \{0, \sqrt{1/S}, -\sqrt{1/S}\}^n : \|x\|_0 \leq S \right\}$$

Repeat for $|\mathcal{X}| = (N/S)^{S/4}$ iterations.

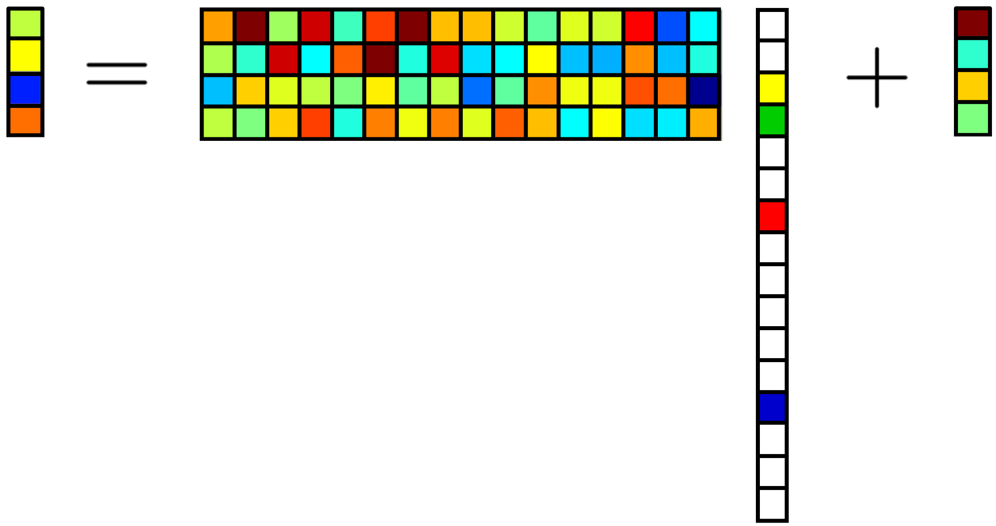
With probability $> 0$, the remaining properties are satisfied.

**Key:** *Matrix Bernstein Inequality* [Ahlswede and Winter, 2002]

# Adaptive Sensing

# Adaptive Sensing

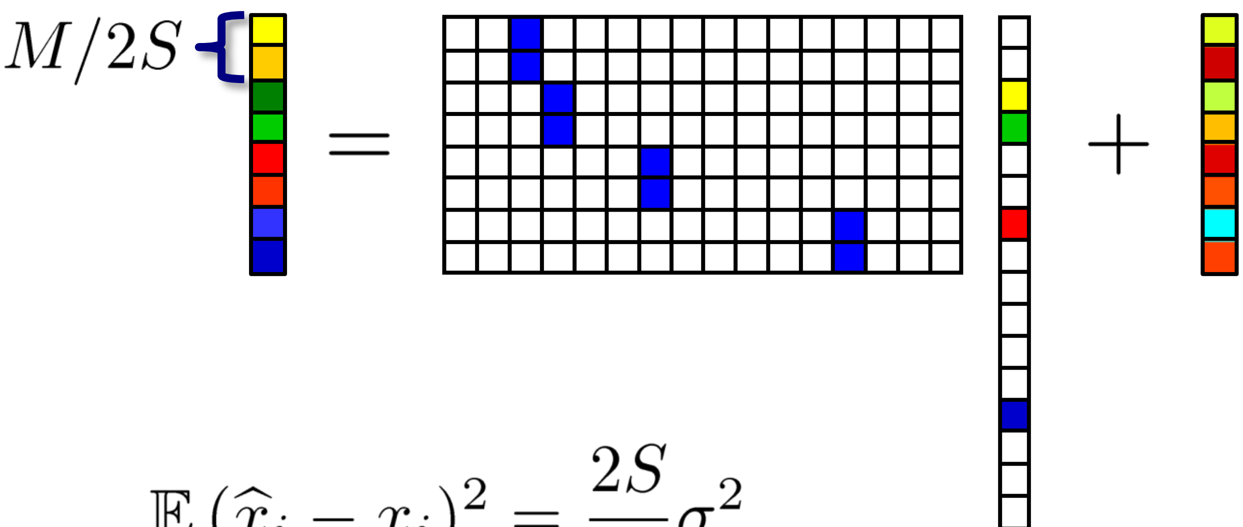
Think of sensing as a game of 20 questions



Simple strategy: Use $M/2$ measurements to find the support, and the remainder to estimate the values.

# Thought Experiment

Suppose that after $M/2$ measurements we have perfectly estimated the support.




$$\mathbb{E}\left(\widehat{x}_i - x_i\right)^2 = \frac{2S}{M}\sigma^2$$

$$\mathbb{E}\,\|\widehat{x} - x\|_2^2 = \frac{2S}{M}S\sigma^2 \ll \frac{N}{M}S\sigma^2 \log N$$

# Does Adaptivity *Really* Help?

Sometimes…

- Noise-free measurements, but non-sparse signal
  - adaptivity doesn't help if you want a uniform guarantee
  - probabilistic adaptive algorithms can reduce the required number of measurements from $O(S\log(N/S))$ to $O(S\log\log(N/S))$ [Indyk et al. – 2011]
- Noisy setting
  - distilled sensing [Haupt et al. – 2007, 2010]
  - adaptivity can reduce the estimation error to

$$\mathbb{E}\,\|\widehat{x}-x\|_2^2 = \frac{N}{M}S\sigma^2$$

$$\mathbb{E}\,\|\widehat{x}-x\|_2^2 = \frac{S}{M}S\sigma^2$$

*Which is it?*

# Which Is It?

Suppose we have a budget of $M$ measurements of the form $y_i = \langle \phi_i, x \rangle + e_i$ where $\|\phi_i\|_2 = 1$ and $e_i \sim \mathcal{N}(0, \sigma^2)$

The vector $\phi_i$ can have an arbitrary dependence on the measurement history, i.e., $(\phi_1, y_1), \ldots, (\phi_{i-1}, y_{i-1})$

**Theorem**

There exist $x$ with $\|x\|_0 \leq S$ such that for *any* adaptive measurement strategy and *any* recovery procedure $\widehat{x}$,

$$\mathbb{E}\, \|\widehat{x}(y) - x\|_2^2 \geq C \frac{N}{M} S \sigma^2.$$

Thus, in general, adaptivity does *not* seem to help!

[Arias-Castro, Candès, and Davenport - 2011]

# Proof Strategy

Step 1: Consider a prior on sparse signals with nonzeros of amplitude $\mu \approx \sigma\sqrt{N/M}$

Step 2: Show that if given a budget of $M$ measurements, you cannot detect the support very well

Step 3: Immediately translate this into a lower bound on the MSE

To make things simpler, we will consider a Bernoulli prior $\pi(x)$ instead of a uniform $S$-sparse prior:

$$x_j = \begin{cases} 0 & \text{with probability } 1 - S/N \\ \mu > 0 & \text{with probability } S/N \end{cases}$$

# Proof of Main Result

Let $T = \{j : x_j \neq 0\}$ and set $\sigma^2 = 1$

For any estimator $\widehat{x}$, define $\widehat{T} := \{j : |\widehat{x}_j| \geq \mu/2\}$

Whenever $j \in T \setminus \widehat{T}$ or $j \in \widehat{T} \setminus T$, $|\widehat{x}_j - x_j| \geq \mu/2$

$$\|\widehat{x} - x\|_2^2 \geq \frac{\mu^2}{4}|T \setminus \widehat{T}| + \frac{\mu^2}{4}|\widehat{T} \setminus T| = \frac{\mu^2}{4}|\widehat{T} \Delta T|$$

$$\Rightarrow \quad \mathbb{E}\,\|\widehat{x} - x\|_2^2 \geq \frac{\mu^2}{4}\mathbb{E}\,|\widehat{T} \Delta T|$$

# Proof of Main Result

**Lemma**

Under the Bernoulli prior, *any* estimate $\widehat{T}$ satisfies

$$\mathbb{E}\,|\widehat{T}\Delta T| \geq S\left(1 - \frac{\mu}{2}\sqrt{\frac{M}{N}}\right).$$

Thus, $\mathbb{E}\,\|\widehat{x} - x\|_2^2 \geq \frac{\mu^2}{4}\mathbb{E}\,|\widehat{T}\Delta T|$

$$\geq S \cdot \frac{\mu^2}{4}\left(1 - \frac{\mu}{2}\sqrt{\frac{M}{N}}\right)$$

Plug in $\mu = \frac{8}{3}\sqrt{\frac{N}{M}}$ and this reduces to

$$\mathbb{E}\,\|\widehat{x} - x\|_2^2 \geq \frac{4}{27} \cdot \frac{SN}{M} \geq \frac{1}{7} \cdot \frac{SN}{M}$$

# Key Ideas in Proof of Lemma

$$\mathbb{P}_{0,j}(y_1,\ldots,y_m) = \mathbb{P}(y_1,\ldots y_m | x_j = 0)$$

$$\mathbb{P}_{1,j}(y_1,\ldots,y_m) = \mathbb{P}(y_1,\ldots,y_m | x_j = \mu)$$

$$\mathbb{E}\,|\widehat{T}\Delta T| \geq \frac{S}{N}\sum_j (1 - \|\mathbb{P}_{1,j} - \mathbb{P}_{0,j}\|_{\mathrm{TV}})$$

$$\geq S - \frac{S}{\sqrt{N}}\sqrt{\sum_j \|\mathbb{P}_{1,j} - \mathbb{P}_{0,j}\|_{\mathrm{TV}}^2}$$

$$\sum_j \|\mathbb{P}_{1,j} - \mathbb{P}_{0,j}\|_{\mathrm{TV}}^2 \leq \frac{\mu^2}{4}M \quad \Rightarrow \quad \mathbb{E}\,|\widehat{T}\Delta T| \geq S\left(1 - \frac{\mu}{2}\sqrt{\frac{M}{N}}\right)$$

# Key Ideas in Proof of Lemma

**Pinsker's Inequality**

$$\|\mathbb{P} - \mathbb{Q}\|_{\mathrm{TV}} \le \sqrt{K(\mathbb{P}, \mathbb{Q})/2}$$

$$\|\mathbb{P}_{1,j} - \mathbb{P}_{0,j}\|_{\mathrm{TV}}^2 \le \frac{\pi_0}{2} K(\mathbb{P}_{0,j}, \mathbb{P}_{1,j}) + \frac{\pi_1}{2} K(\mathbb{P}_{1,j}, \mathbb{P}_{0,j})$$

$$\le \frac{\mu^2}{4} \sum_i \mathbb{E}\, \phi_{i,j}^2$$

$$\Rightarrow \sum_j \|\mathbb{P}_{1,j} - \mathbb{P}_{0,j}\|_{\mathrm{TV}}^2 \le \frac{\mu^2}{4} \sum_{i,j} \mathbb{E}\, \phi_{i,j}^2 = \frac{\mu^2}{4} M$$

# Adaptivity in Practice

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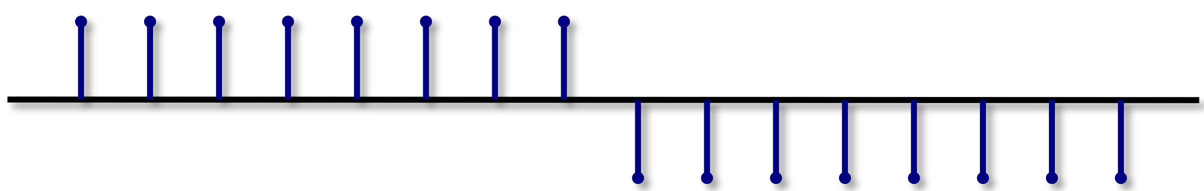
Suppose that $S = 1$ and that $x_{j^*} = \mu$

Binary Search [Iwen and Tewfik – 2011, Davenport and Arias-Castro – 2012]

- split measurements into $\log N$ stages
- in each stage, use measurements to decide if the nonzero is in the left or right half of the "active set"
- after subdividing $\log N$ times, return support



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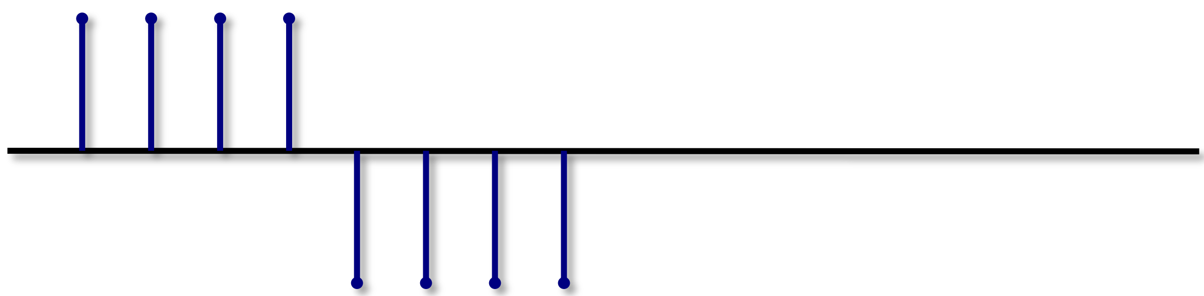

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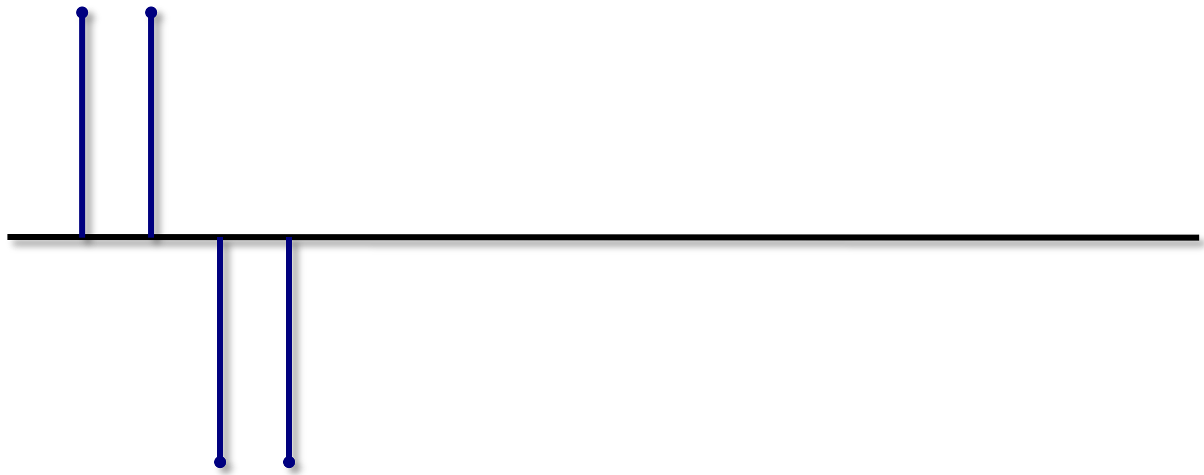

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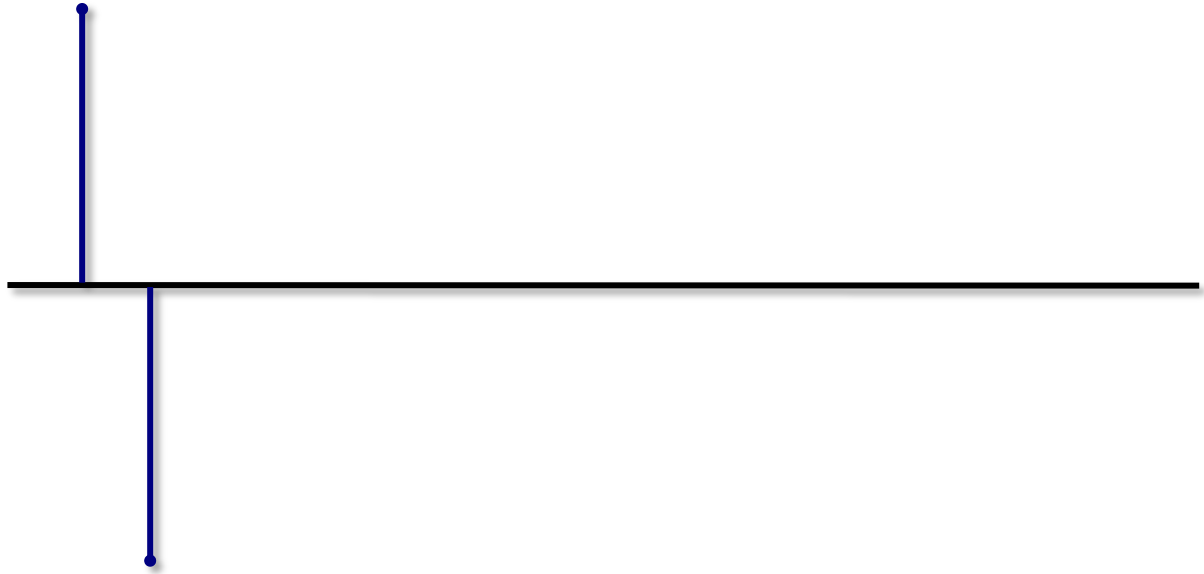
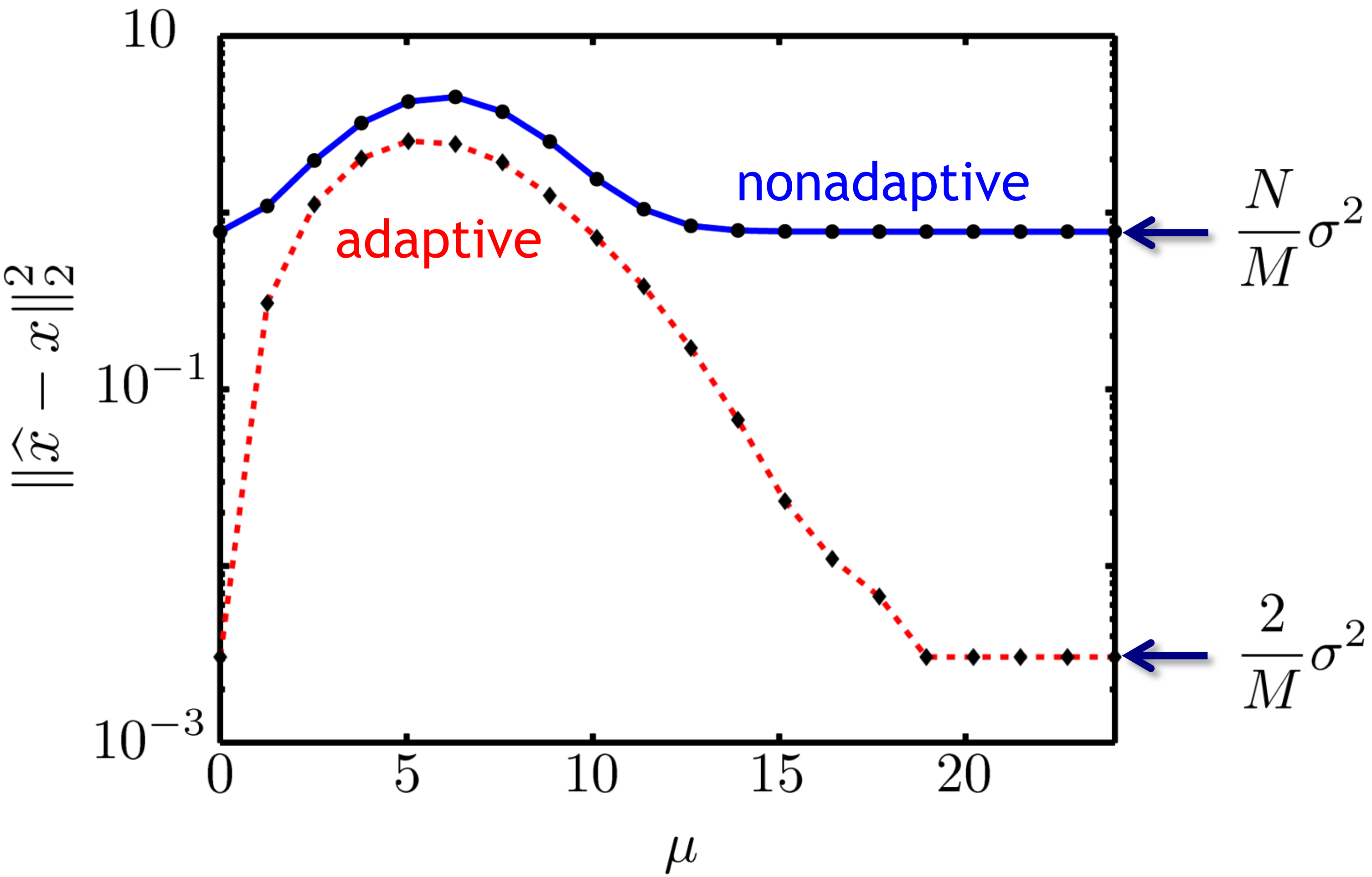


# Experimental Results




$N = 512$

$M = 256$

[Arias-Castro, Candès, and Davenport - 2011]

# Open Questions

- No method can succeed when $\frac{\mu}{\sigma} \approx \sqrt{N/M}$, but the binary search approach succeeds as long as $\frac{\mu}{\sigma} \geq C\sqrt{N/M}$

  [Davenport and Arias-Castro; Malloy and Nowak - 2012]

- Practical algorithms that work well for all values of $\mu$

- Optimal algorithms for $S > 1$

- New theory for restricted adaptive measurements
  - single-pixel camera: 0/1 measurements
  - magnetic resonance imaging (MRI): Fourier measurements
  - analog-to-digital converters: linear filter measurements

- New sensors and architectures that can actually acquire adaptive measurements

# More Information

http://stat.stanford.edu/~markad

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