Sparse Geodesic Paths

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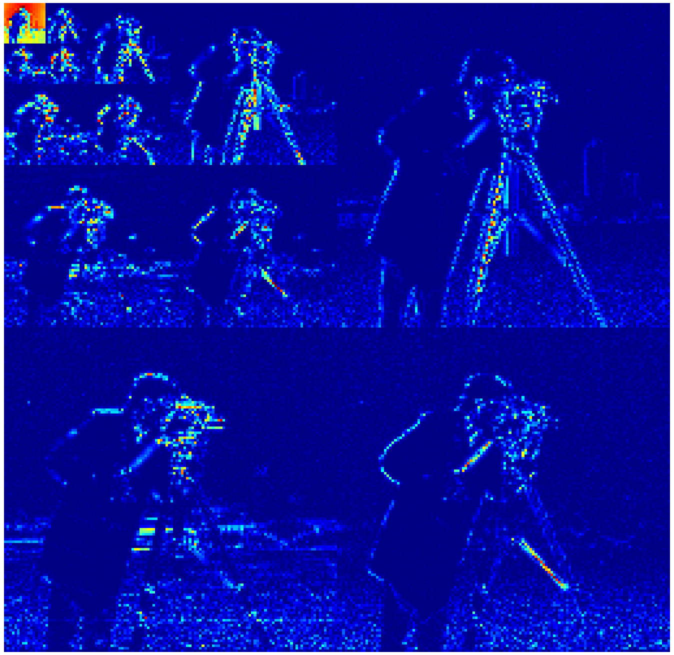
Sparse Signals

Basis transformation

DCT, wavelets



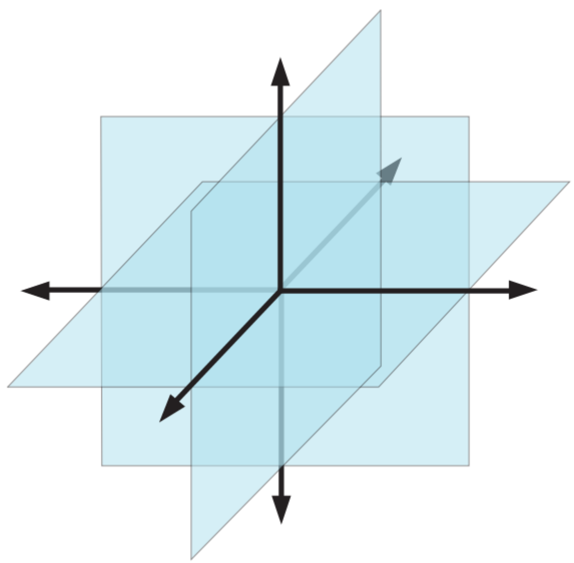
Sparse: $K \ll N$ *nonzero* coefficients

Compressible: $K \ll N$ *important* coefficients

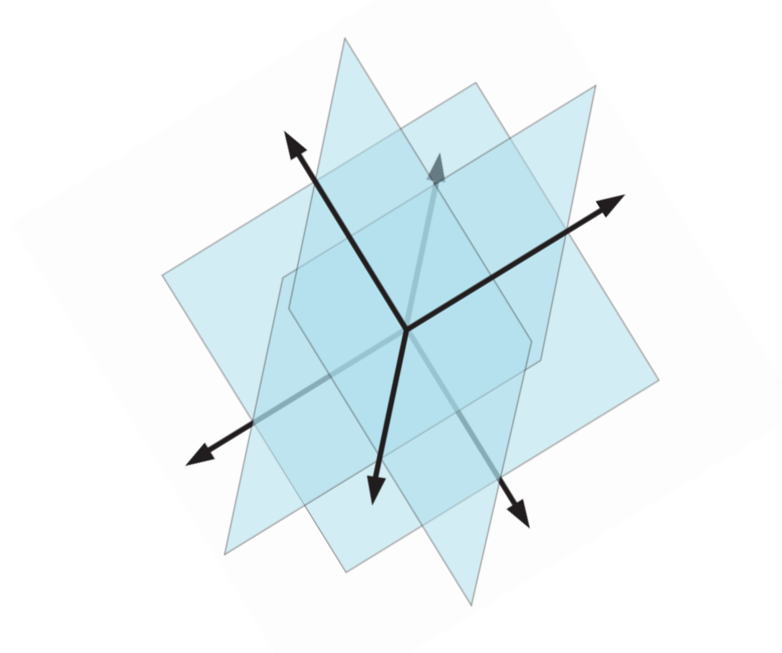
Unions of Subspaces

- Sparse signal ≠ subspaces
  - subspace model: linear
  - sparse model: nonlinear
  - sparse model = union of $\binom{N}{K}$ subspaces

$$\Sigma_K := \{x : \|x\|_0 \leq K\}$$

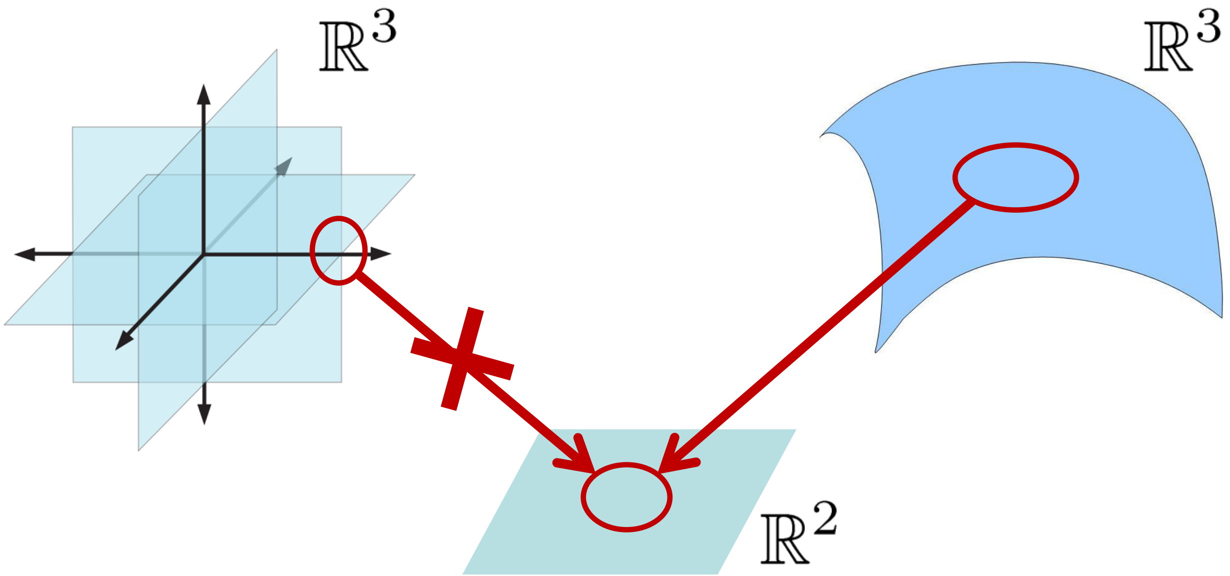


$$\Psi(\Sigma_K) := \{\Psi x : \|x\|_0 \leq K\}$$



Sparsity vs Manifolds

- Does the set of sparse signals form a manifold?

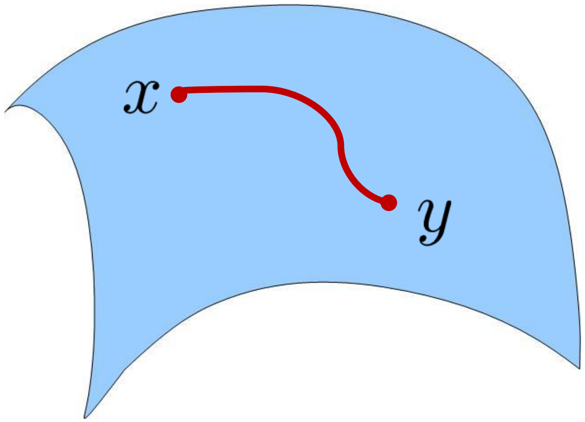


- Union of multiple manifolds
- Same lessons apply – we can still exploit the low-dimensional structure

Geodesic Paths on a Manifold

- How is manifold structure exploited in practice?
- Replace Euclidean distance with *geodesic distance*

$$\phi : [0,1] \to \mathcal{X}$$



$$\Phi_{\mathcal{X}}(x,y) = \{\phi(t) : \phi(0) = x, \phi(1) = y, \phi(t) \in \mathcal{X}\}$$

Geodesic path

$$\gamma = \arg\inf_{\phi \in \Phi_{\mathcal{X}}(x,y)} L(\phi)$$

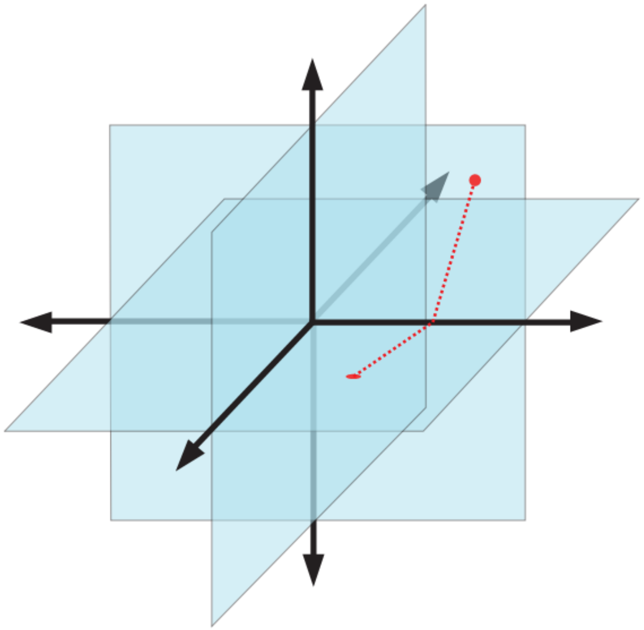
Geodesic distance

$$d_{\mathcal{X}}(x,y) = L(\gamma)$$

Sparse Geodesic Paths

$$\gamma = \arg\inf_{\phi \in \Phi_{\Sigma_K}(x,y)} L(\phi)$$

$$d_{\Sigma_K}(x, y) = L(\gamma)$$



- Assumptions
  - $\Psi = I$
  - $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$
  - $|\mathrm{supp}(x)| = |\mathrm{supp}(y)| = K$

Necessary Conditions

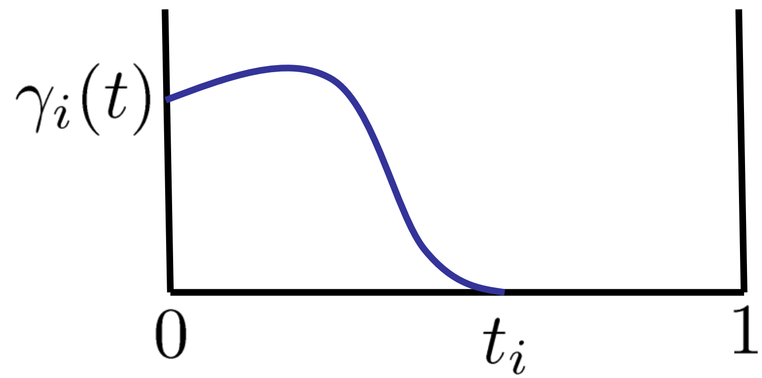
Three cases:

- $i \notin \mathrm{supp}(x) \cup \mathrm{supp}(y)$ $\Rightarrow$ $\gamma_i(t) = 0 \;$ for all $\; t \in [0,1]$

- $i \in \mathrm{supp}(x)$ $\Rightarrow$

  $\exists \; t_i$ such that

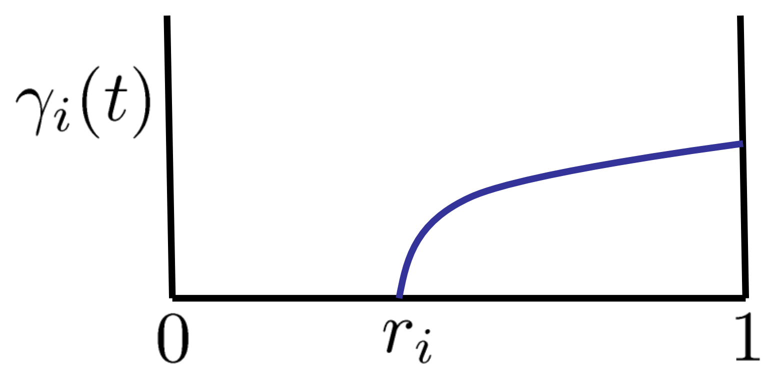
  $\gamma_i(t) = 0 \;$ for all $t \in [t_i, 1]$



- $i \in \mathrm{supp}(y)$ $\Rightarrow$

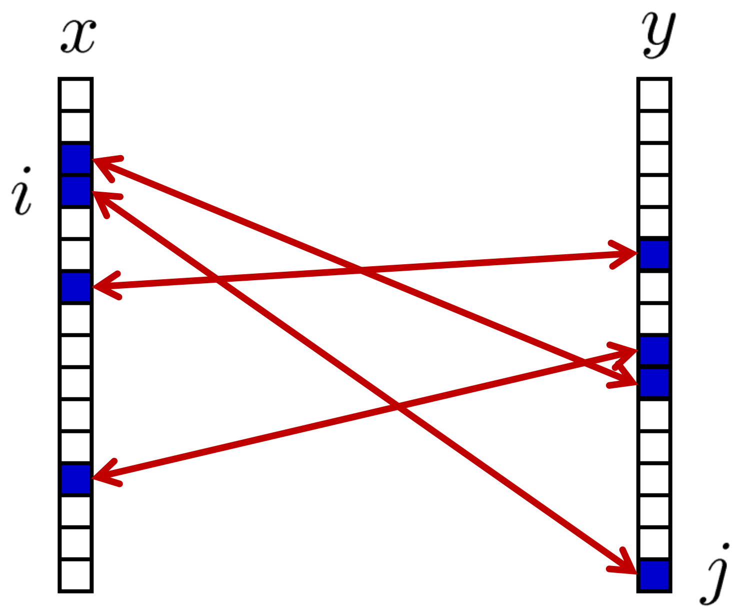
  $\exists \; r_i$ such that

  $\gamma_i(t) = 0 \;$ for all $t \in [0, r_i]$



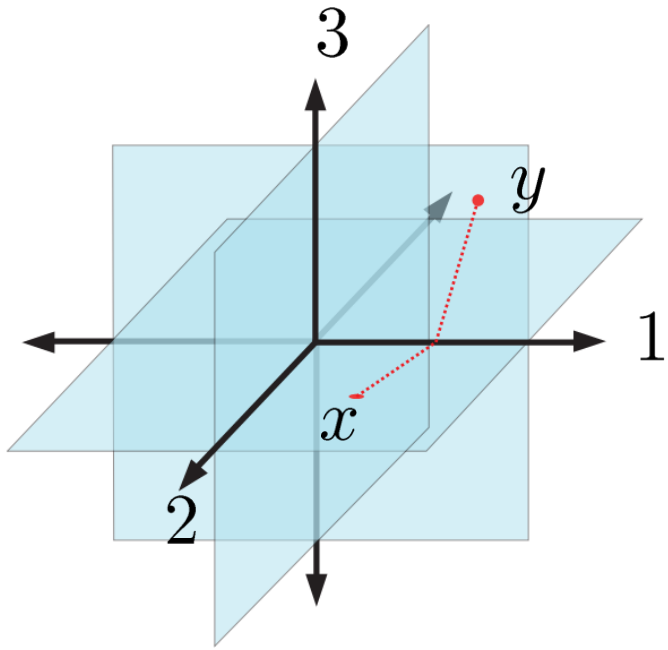
Support Matching

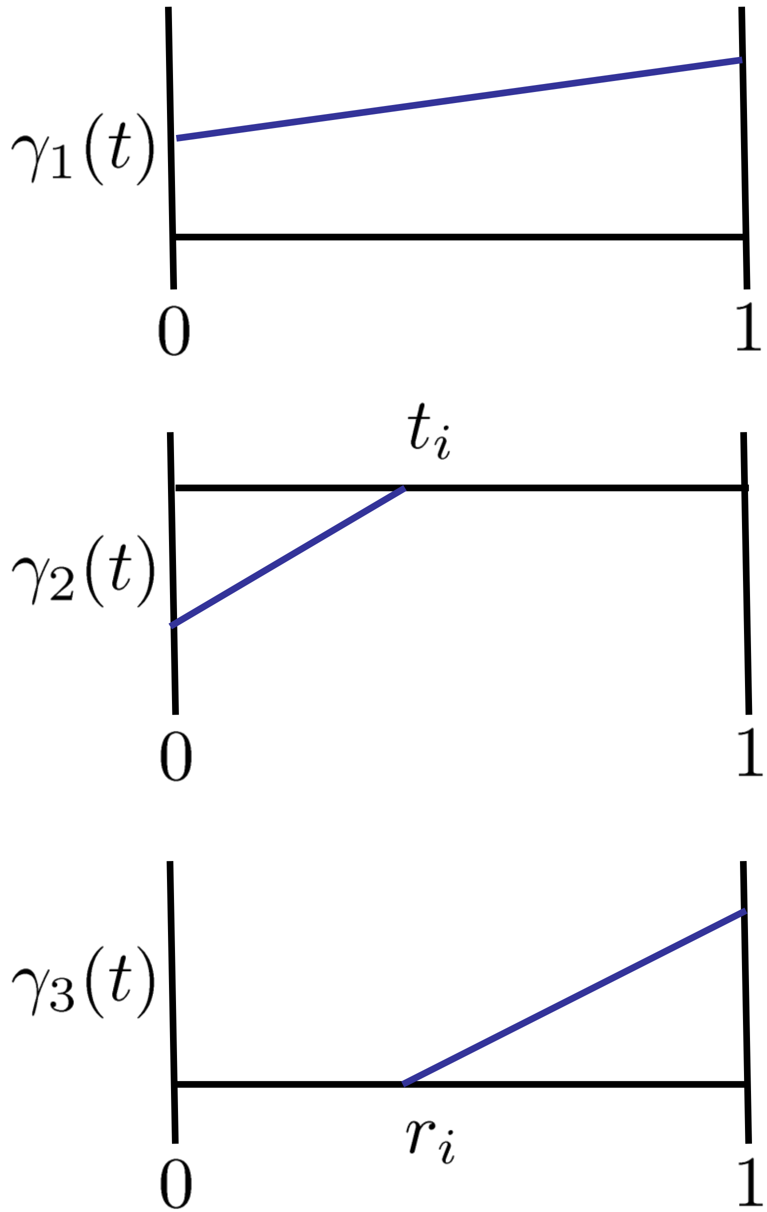
- Given a candidate $\gamma(t)$, we can define a *matching* $\mathcal{M}$ between the entries of x and y



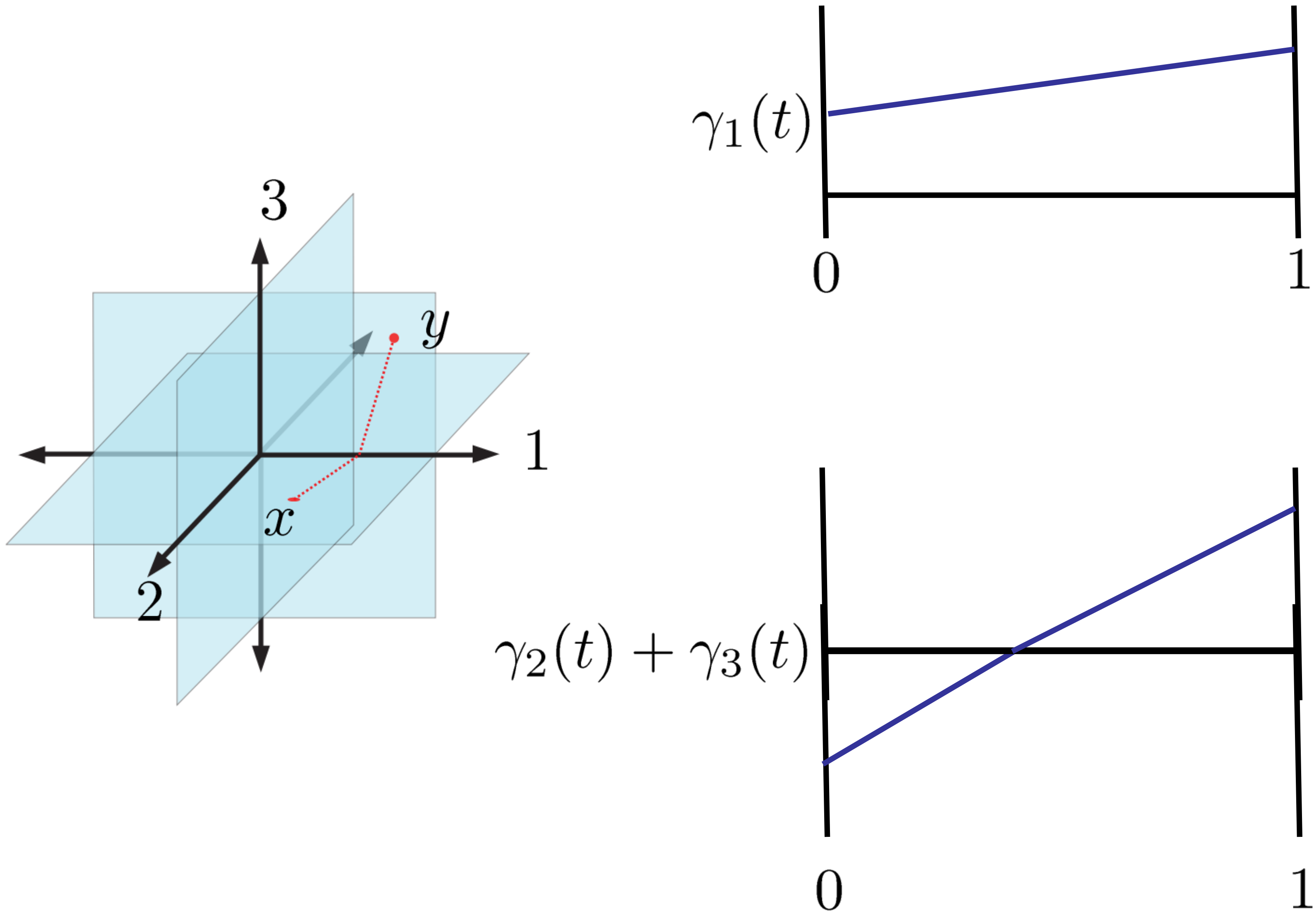
- We allow $(i, j) \in \mathcal{M}$ if and only if $t_i \leq r_j$

Geodesic "Unfolding"



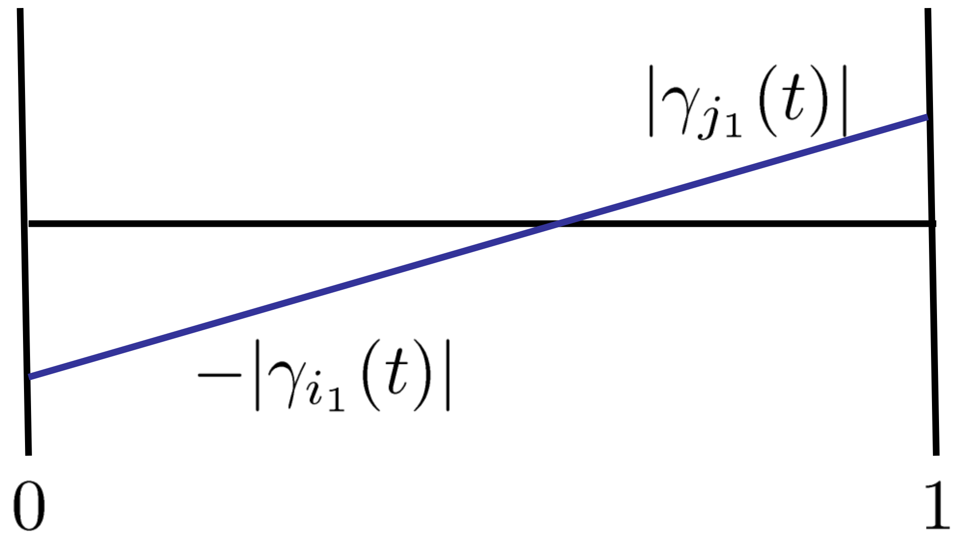


Geodesic “Unfolding”



Geodesic "Unfolding"

$$(i_1, j_1) \in \mathcal{M}$$



- Repeating for every $(i_k, j_k) \in \mathcal{M}$, we can map any candidate geodesic $\gamma(t)$ into a path in $\mathbb{R}^K$ from $-|\gamma_I(0)|$ to $|\gamma_J(1)|$

Sketch of Derivation

1. Any potential geodesic path is compatible with at least one matching

2. Given any potential geodesic path, its length is equal to the length of the corresponding "unfolded" path

3. Given any matching, the shortest path in the "unfolded" space is a straight line

4. This line defines a valid geodesic path

Matching Dependent Geodesic

- Given a matching $\mathcal{M}$, the shortest path compatible with this matching has length

$$\sqrt{\sum_{k=1}^{K} \left(|x_{i_k}| + |y_{j_k}|\right)^2}$$

- Finding the shortest path is equivalent to finding the best matching

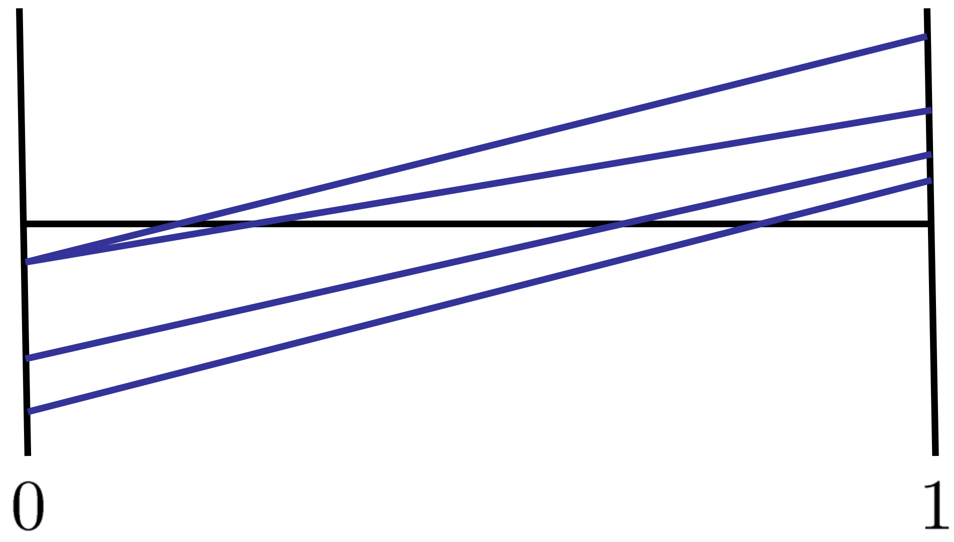
Optimal Matching

- We want to minimize

$$\sum_{k=1}^{K} \left(|x_{i_k}| + |y_{j_k}|\right)^2 = \|x - y\|_2^2 + 2\sum_{k=1}^{K} |x_{i_k}||y_{j_k}|$$

- Set $|x_{i_1}| \leq |x_{i_2}| \leq \cdots \leq |x_{i_K}|$

$$|y_{j_1}| \geq |y_{j_2}| \geq \cdots \geq |y_{j_K}|$$



Observations

- Attempts to equalize the value of each term in the sum

$$d_{\Sigma_K}(x,y) = \sqrt{\|x-y\|_2^2 + 2\sum_{k=1}^{K} |x_{i_k}||y_{j_k}|}$$

- Assume $x_{i_k} = C_x$ and $y_{j_k} = C_y$

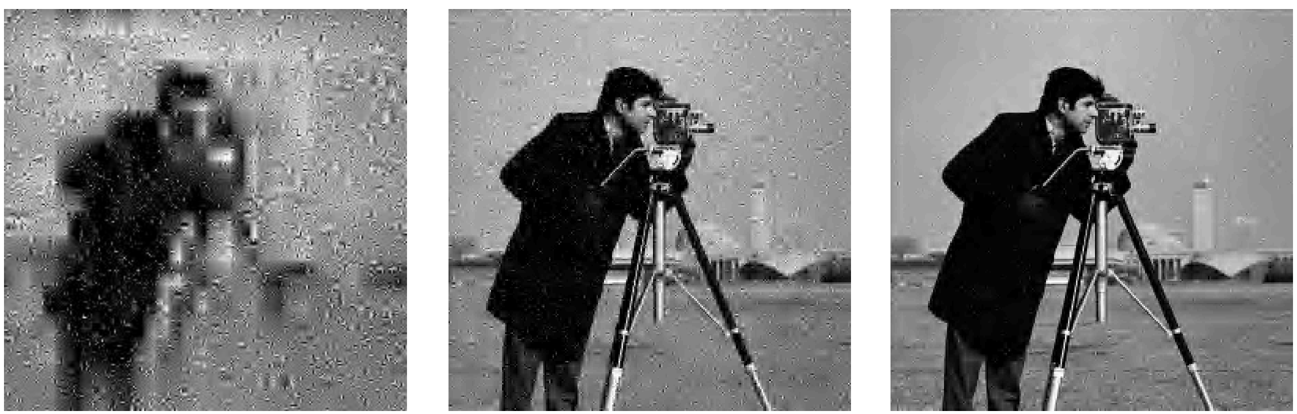
$$\sum_{k=1}^{K} |x_{i_k}||y_{j_k}| = KC_xC_y = \|x\|_2\|y\|_2$$

$$\|x-y\|_2 \leq d_{\Sigma_K}(x,y) \leq \|x\|_2 + \|y\|_2$$

Example

$$d_{\Sigma_K}(x, x+n) = \|n\|_2$$

$$d_{\Sigma_K}(x, x+n) > \|n\|_2$$



10 dB 20 dB 30 dB

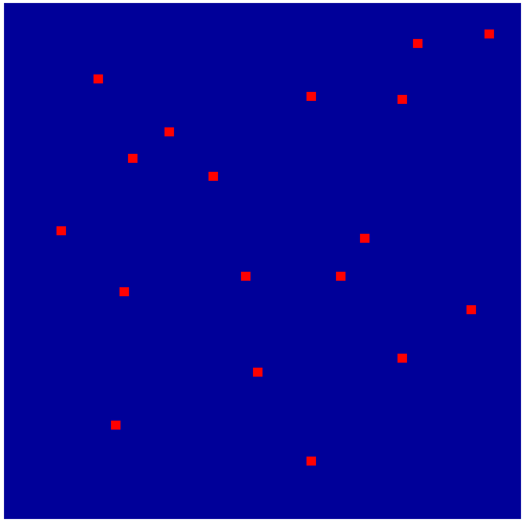
SNR

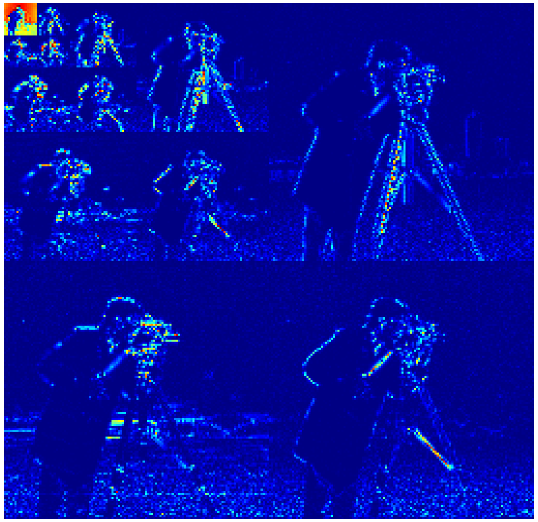
What is it good for?

- Incorporating prior knowledge
  - use geodesic distance as input to kNN, SVM, or other kernel-based learning algorithm
- Semi-supervised learning
  - combine with dictionary learning algorithms such as K-SVD [Aharon 2006]
- Signal morphing/interpolation
- "Absolutely nothin'!"? [Starr 1970]

Extensions

- Structured sparsity





- Compressible data
  - truncate to enforce sparsity
  - geodesic distance on ℓ_p and/or $w\ell_p$ balls

Conclusions

- For the simple sparse setting
  - analytic formula available
  - doesn't differ much from Euclidean distance
- Important to incorporate additional structure/models
  - still possible to derive a formula?
  - can it be computed efficiently?
- Promising applications?

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