Dr. Leach's Filter Potpourri

Transfer Functions

For sinusoidal time variations, the input voltage to a filter can be written

$$v_I(t) = \text{Re}\left[V_i e^{j\omega t}\right]$$

where V_i is the phasor input voltage, i.e. it has an amplitude and a phase, and $e^{j\omega t} = \cos \omega t + j \sin \omega t$. A sinusoidal signal is the only signal in nature that is preserved by a linear system. Therefore, if the filter is linear, its output voltage can be written

$$v_O(t) = \text{Re}\left[V_o e^{j\omega t}\right]$$

where V_o is the phasor output voltage. The ratio of V_o to V_i is called the voltage-gain transfer function. It is a function of frequency. Let us denote

$$T(j\omega) = \frac{V_o}{V_i}$$

We can write $T(j\omega)$ as follows:

$$T(j\omega) = A(\omega) e^{j\varphi(\omega)}$$

where $A(\omega)$ and $\varphi(\omega)$ are real functions of ω. $A(\omega)$ is called the gain function and $\varphi(\omega)$ is called the phase function.

As an example, consider the filter input voltage

$$v_I(t) = V_1 \cos(\omega t + \theta) = \text{Re}\left[V_1 e^{j\theta} e^{j\omega t}\right]$$

The corresponding phasor input and output voltages are

$$V_i = V_1 e^{j\theta}$$

$$V_o = V_1 e^{j\theta} A(\omega) e^{j\varphi(\omega)}$$

It follows that the filter voltage is

$$v_O(t) = \text{Re}\left[V_1 e^{j\theta} A(\omega) e^{j\varphi(\omega)} e^{j\omega t}\right] = A(\omega) V_1 \cos[\omega t + \theta + \varphi(\omega)]$$

This equation illustrates why $A(\omega)$ is called the gain function and $\varphi(\omega)$ is called the phase function.

The complex frequency s is usually used in place of $j\omega$ in writing transfer functions. In general, most transfer functions can be written in the form

$$T(s) = K\frac{N(s)}{D(s)}$$

where K is a gain constant and $N(s)$ and $D(s)$ are polynomials in s containing no reciprocal powers of s. The roots of $D(s)$ are called the poles of the transfer function. The roots of $N(s)$ are called the zeros. As an example, consider the function

$$T(s) = 4\frac{s/4 + 1}{s^2/6 + 5s/6 + 1} = 4\frac{s/4 + 1}{(s/2 + 1)(s/3 + 1)}$$

The function has a zero at $s = -4$ and poles at $s = -2$ and $s = -3$. Note that $T(\infty) = 0$. Because of this, some texts would say that $T(s)$ has a zero at $s = \infty$. However, this is not correct because $N(\infty) \neq 0$.

Note that the constant terms in the numerator and denominator of $T(s)$ are both unity. This is one of two standard ways for writing transfer functions. Another way is to make the coefficient of the highest powers of s unity. In this case, the above transfer function would be written

$$T(s) = 6\frac{s + 4}{s^2 + 5s + 6} = 6\frac{s + 4}{(s + 2)(s + 3)}$$

Because it is usually easier to construct Bode plots with the first form, that form is used here.

Because the complex frequency s is the operator which represents d/dt in the differential equation for a system, the transfer function contains the differential equation. Let the transfer function above represent the voltage gain of a circuit, i.e. $T(s) = V_o/V_i$, where V_o and V_i, respectively, are the phasor output and input voltages. It follows that

$$\left(\frac{s^2}{6} + \frac{5s}{6} + 1\right) V_o = 4\left(\frac{s}{4} + 1\right) V_i$$

When the operator s is replaced with d/dt, the following differential equation is obtained:

$$\frac{1}{6}\frac{d^2 v_O}{dt^2} + \frac{5}{6}\frac{dv_O}{dt} + v_O = \frac{dv_I}{dt} + 4v_I$$

where v_O and v_I, respectively, are the time domain output and input voltages. Note that the poles are related to the derivatives of the output and the zeros are related to the derivatives of the input.

How to Do Bode Plots

A Bode plot is a plot of either the magnitude or the phase of a transfer function $T(j\omega)$ as a function of ω. The magnitude plot is the more common plot because it represents the gain of the system. Therefore, the term "Bode plot" usually refers to the magnitude plot. The rules for making Bode plots can be derived from the following transfer function:

$$T(s) = K\left(\frac{s}{\omega_0}\right)^{\pm n}$$

where n is a positive integer. For $+n$ as the exponent, the function has n zeros at $s = 0$. For $-n$, it has n poles at $s = 0$. With $s = j\omega$, it follows that $T(j\omega) = Kj^{\pm n}(\omega/\omega_0)^{\pm n}$, $|T(j\omega)| = K(\omega/\omega_0)^{\pm n}$ and $\angle T(j\omega) = \pm n \times 90°$. If ω is increased by a factor of 10, $|T(j\omega)|$ changes by a factor of $10^{\pm n}$. Thus a plot of $|T(j\omega)|$ versus ω on $\log - \log$ scales has a slope of $\log(10^{\pm n}) = \pm n$ decades/decade. There are $20\,\mathrm{dBs}$ in a decade, so the slope can also be expressed as $\pm 20n\,\mathrm{dB/decade}$.

As a first example, consider the low-pass transfer function

$$T(s) = \frac{K}{1 + s/\omega_1}$$

This function has a pole at $s = -\omega_1$ and no zeros. For $s = j\omega$ and $\omega/\omega_1 \ll 1$, we have $T(j\omega) \simeq K$, $|T(j\omega)| \simeq K$, and $\angle T(j\omega) \simeq 0 \times 90° = 0°$. For $\omega/\omega_1 \gg 1$, $T(j\omega) \simeq K(j\omega/\omega_1)^{-1}$, $|T(j\omega)| \simeq K(\omega/\omega_1)^{-1}$, and $\angle T(j\omega) \simeq -1 \times 90° = -90°$. On $\log - \log$ scales,

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the magnitude plot for the low-frequency approximation has a slope of 0 while that for the high-frequency approximation has a slope of -1. The low and high-frequency approximations intersect when $K = K(\omega_1/\omega)$, or when $\omega = \omega_1$. For $\omega = \omega_1$, $|T(j\omega)| = K/|1+j| = K/\sqrt{2}$ and $\angle T(j\omega) = -\arctan(1) = -45°$. Note that this is the average value of the phase on the two adjoining asymptotes. The Bode magnitude and phase plots are shown in Fig. 1. Note that the slope of the asymptotic magnitude plot rotates by -1 at $\omega = \omega_1$. Because ω_1 is the magnitude of the pole frequency, we say that the slope rotates by -1 at a pole. A straight line segment that is tangent to the phase plot at $\omega = \omega_1$ would intersect the $0°$ level at $\omega_1/4.81$ and the $-90°$ level at $4.81\omega_1$.
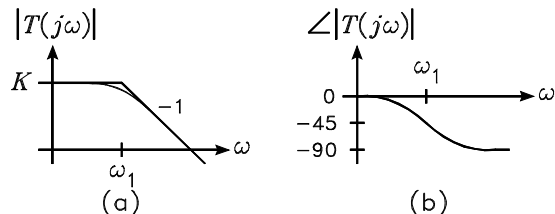

Figure 1   Bode plots. (a) Magnitude. (b) Phase.

As a second example, consider the transfer function

$$T(s) = K\left(1 + \frac{s}{\omega_1}\right)$$

This function has a zero at $s = -\omega_1$. For $s = j\omega$ and $\omega/\omega_1 \ll 1$, we have $T(j\omega) \simeq K$, $|T(j\omega)| \simeq K$, and $\angle T(j\omega) \simeq 0 \times 90° = 0°$. For $\omega/\omega_1 \gg 1$, $T(j\omega) \simeq K(j\omega/\omega_1)^1$ $|T(j\omega)| \simeq K(\omega/\omega_1)$ and $\angle T(j\omega) \simeq +1\times90° = 90°$. On $\log-\log$ scales, the magnitude plot for the low-frequency approximation has a slope of 0 while that for the high-frequency approximation has a slope of $+1$. The low and high-frequency approximations intersect when $K = K(\omega/\omega_1)$, or when $\omega = \omega_1$. For $\omega = \omega_1$, $|T(j\omega)| = K\sqrt{2}$ and $\angle T(j\omega) = \arctan(1) = 45°$. Note that this is the average of the phase on the two adjoining asymptotes. The Bode magnitude and phase plots are shown in Fig. 2. Note that the slope of the asymptotic magnitude plot rotates by $+1$ at $\omega = \omega_1$. Because ω_1 is the magnitude of the zero frequency, we say that the slope rotates by $+1$ at a zero. A straight line segment that is tangent to the phase plot at $\omega = \omega_1$ would intersect the $0°$ level at $\omega_1/4.81$ and the $90°$ level at $4.81\omega_1$.
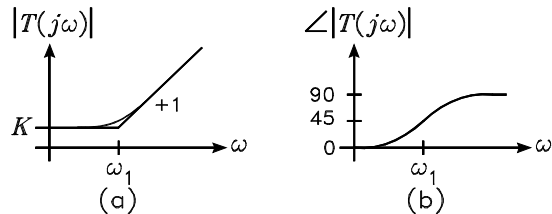

Figure 2   Bode plots. (a) Magnitude. (b) Phase.

From the above examples, we can summarize the basic rules for making Bode plots as follows:

1. In any frequency band where a transfer function can be approximated by $K(j\omega/\omega_0)^{\pm n}$, the slope of the Bode magnitude plot is $\pm n$ dec/dec. The phase is $\pm n \times 90°$.

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2. Poles cause the asymptotic slope of the magnitude plot to rotate clockwise by one unit at the pole frequency.

3. Zeros cause the asymptotic slope of the magnitude plot to rotate counter-clockwise by one unit at the zero frequency.

As a third example, consider the transfer function

$$T\left(s\right) = K\frac{s/\omega_1}{s/\omega_1 + 1}$$

This function has a pole at $s = -\omega_1$ and a zero at $s = 0$. For $s = j\omega$ and $\omega/\omega_1 \ll 1$, we have $|T\left(j\omega\right)| \simeq K\left(\omega/\omega_1\right)$ and $\angle T\left(j\omega\right) \simeq 90\,°$. For $\omega/\omega_1 \gg 1$, $|T\left(j\omega\right)| \simeq K$ and $\angle T\left(j\omega\right) \simeq 0\,°$. On $\log - \log$ scales, the magnitude plot for the low-frequency approximation has a slope of $+1$ while that for the high-frequency approximation has a slope of 0. The low and high-frequency approximations intersect when $K\left(\omega/\omega_1\right) = K$, or when $\omega = \omega_1$. For $\omega = \omega_1$, $|T\left(j\omega\right)| = K/\sqrt{2}$ and $\angle T\left(j\omega\right) = 90\,° - \arctan\left(1\right) = 45\,°$. The Bode magnitude and phase plots are shown in Fig. 3. Note that the slope of the asymptotic magnitude plot rotates by -1 at the pole. The transfer function is called a high-pass function because its gain approaches zero at low frequencies.
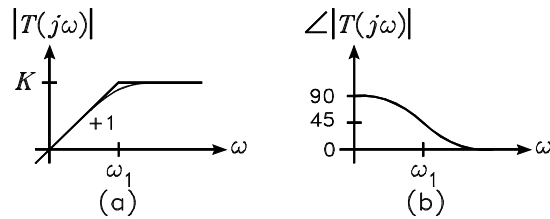

Figure 3   Bode plots. (a) Magnitude. (b) Phase.

A shelving transfer function has the form

$$T\left(s\right) = K\frac{1 + s/\omega_2}{1 + s/\omega_1}$$

The function has a pole at $s = -\omega_1$ and a zero at $s = -\omega_2$. We will consider the low-pass shelving function for which $\omega_1 < \omega_2$. For $s = j\omega$ and $\omega/\omega_1 \ll 1$, we have $|T\left(j\omega\right)| \simeq K$ and $\angle T\left(j\omega\right) \simeq 0\,°$. As ω is increased, the pole causes the asymptotic slope to rotate from 0 to -1 at ω_1. The zero causes the asymptotic slope to rotate from -1 back to 0 at ω_2. For $\omega/\omega_2 \gg 1$, $|T\left(j\omega\right)| \simeq K\left(\omega_1/\omega_2\right)$. The Bode magnitude plot is shown in Fig. 4(a). If the transfer function did not have the zero, the actual gain at ω_1 would be $K/\sqrt{2}$. The zero causes the gain to be between $K/\sqrt{2}$ and K. Similarly, the pole causes the actual gain at ω_2 to be between $K\left(\omega_1/\omega_2\right)$ and $\sqrt{2}K\left(\omega_1/\omega_2\right)$. The actual plot intersects the asymptotic plot at the geometric mean frequency $\sqrt{\omega_1\omega_2}$.

The phase plot has a slope that approaches $0\,°$ at very low frequencies and at very high frequencies. At the geometric mean frequency $\sqrt{\omega_1\omega_2}$, the phase is approaching $-90\,°$. If the function only had a pole, the phase at ω_1 would be $-45\,°$, approaching $-90\,°$ at higher frequencies. However, the zero causes the high-frequency phase to approach $0\,°$. Thus the phase at ω_1 is more positive than $-45\,°$. At the geometric mean frequency $\sqrt{\omega_1\omega_2}$, the slope of the phase function is zero. The Bode phase plot is shown in Fig. 4(b).
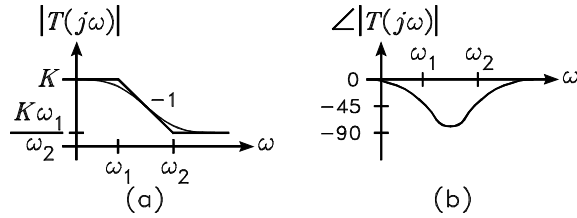
Figure 4   Bode plots. (a) Magnitude. (b) Phase.

Classes of Filter Functions

The filters considered in this experiment can be divided into four classes. These are low-pass, high-pass, band-pass and band-reject. Although it is impossible to realize an ideal filter, the characteristics of the four classes of filters are simplest to describe for ideal filters. An ideal low-pass filter has a cutoff frequency below which the gain is independent of frequency and above which the gain is zero. Fig. 5(a) illustrates the magnitude response of an ideal low-pass filter having a gain K. The responses of two physically realizable filters are also shown. An ideal high-pass filter has a cutoff frequency above which the gain is constant and below which the gain is zero. The magnitude responses of an ideal high-pass filter and two physically realizable filters are illustrated in Fig. 5(b). An ideal band-pass filter has two cutoff frequencies between which the gain is constant and zero elsewhere. The magnitude responses of an ideal band-pass filter and two physically realizable filters are illustrated in Fig. 5(c). An ideal band-reject filter has two cutoff frequencies between which the gain is zero and constant elsewhere. The magnitude responses of an ideal band-reject filter and two physically realizable filters are illustrated in Fig. 5(d).
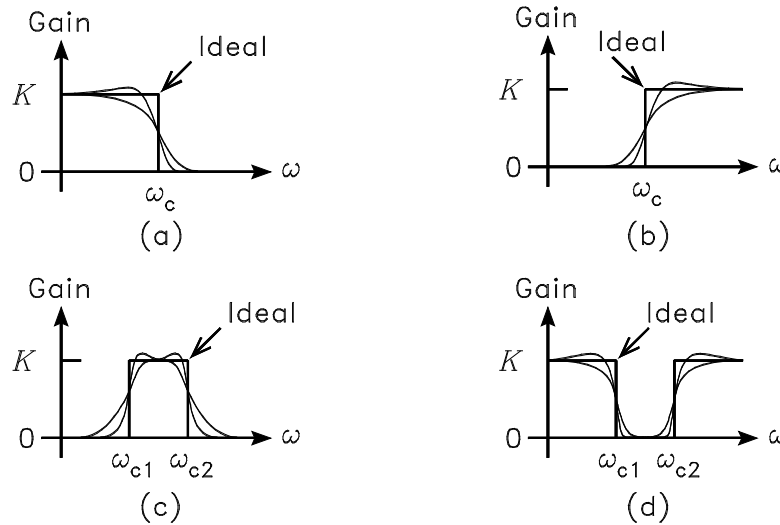

Figure 5   (a) Low pass. (b) High pass. (c) Band pass. (d) Band reject.

Low-pass filters are used in applications where it is desired to remove the high-frequency content of a signal. For example, aliasing distortion can occur if a signal is applied to the input of an analog-to-digital converter that has a frequency higher than one-half the sampling frequency of the converter. A low-pass filter might be used to limit the bandwidth of the signal. Similarly, a high-pass filter is used in applications where it is desired to remove the low-frequency content of a signal. For example, the tweeter driver in a loudspeaker can be

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damaged by low frequencies signals. To prevent this, a high-pass filter called a crossover network must be connected to the tweeter.

A band-pass filter is used in applications where it is desired to pass only the frequencies in a band. For example, to detect a low level tone that is buried in noise, a band-pass filter might be used to pass the tone and reject the noise. A band-reject filter is used in applications where it is desired to reject a particular frequency or band of frequencies. For example, a 60 Hz hum induced in the amplifier of a public address system might be filtered out with a band-pass filter.

Frequency Transformations

Filter transfer functions are normally derived as low-pass functions. Frequency transformations are then used to transform the low-pass functions into either high-pass, band-pass, or band-reject transfer functions. For a low-pass filter, let the normalized frequency p be defined by

$$p = \frac{s}{\omega_c}$$

where ω_c is a normalization frequency. For the case of low-pass and high-pass filters, ω_c is the called the cutoff frequency of the filter. Depending on the type of filter, it is not necessarily the $-3\,\mathrm{dB}$ cutoff frequency. To distinguish between the two in the following, ω_3 is used to denote the $-3\,\mathrm{dB}$ cutoff frequency. In the case of a band-pass filter, ω_c is the center frequency of the band-pass response. In the case of a band-reject filter, ω_c is the center frequency of the band-reject response.

The frequency transformations are defined as follows:

$$\text{Low-Pass to High-Pass} \quad p \;\rightarrow\; \frac{1}{p}$$

$$\text{Low-Pass to Band-Pass} \quad p \;\rightarrow\; B\left(p + \frac{1}{p}\right)$$

$$\text{Low-Pass to Band-Reject} \quad p \;\rightarrow\; B^{-1}\left(p + \frac{1}{p}\right)^{-1}$$

where the arrow is read "is replaced by". The parameter B determines the $-3\,\mathrm{dB}$ bandwidth of the band-pass and band-reject functions.

Transformations of First-Order Functions

As an example, consider the first-order low-pass filter function

$$T_{LP}(s) = \frac{1}{1 + s/a\omega_c}$$

where a is a positive constant. The function $T_{LP}(p)$ is given by

$$T_{LP}(p) = \frac{1}{1 + p/a}$$

The high-pass, band-pass, and band-reject transfer functions are given by

$$\text{High-Pass} \quad T_{HP}(p) = \frac{1}{1 + 1/ap} = \frac{ap}{1 + ap}$$

$$\text{Band-Pass} \quad T_{BP}(p) = \frac{1}{1 + B(p + 1/p)/a} = \frac{(a/B)\,p}{p^2 + (a/B)\,p + 1}$$

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$$\text{Band-Reject} \qquad T_{BR}(p) = \frac{1}{1 + B^{-1}(p + 1/p)^{-1}/a} = \frac{p^2 + 1}{p^2 + (1/aB)p + 1}$$

Note that the order of the transfer function is doubled for the band-pass and band-reject transformations.

Transformations of Second-Order Functions

Consider the second-order low-pass function

$$T_{LP}(s) = \frac{1}{(s/a\omega_c)^2 + (1/b)(s/a\omega_c) + 1}$$

where a and b are positive constants. The function $T_{LP}(p)$ is given by

$$T_{LP}(p) = \frac{1}{(p/a)^2 + (1/b)(p/a) + 1}$$

The high-pass, band-pass, and band-reject transfer functions are given by

$$\text{High Pass} \qquad T_{HP}(p) = \frac{1}{(1/ap)^2 + (1/b)(1/ap) + 1} = \frac{(ap)^2}{(ap)^2 + (1/b)(ap) + 1}$$

$$
\begin{aligned}
\text{Band Pass} \qquad T_{BP}(p) &= \frac{1}{[B(p + 1/p)/a]^2 + (1/b)[B(p + 1/p)/a] + 1} \\
&= \frac{(a^2/B^2)p^2}{p^4 + (a/bB)p^3 + (a^2/B^2 + 2)p^2 + (a/bB)p + 1}
\end{aligned}
$$

$$
\begin{aligned}
\text{Band Reject} \qquad T_{BR}(p) &= \frac{1}{\left[B^{-1}(p + 1/p)^{-1}/a\right]^2 + (1/b)\left[B^{-1}(p + 1/p)^{-1}/a\right] + 1} \\
&= \frac{(p^2 + 1)^2}{p^4 + (1/abB)p^3 + (1/a^2 B^2 + 2)p^2 + (1/abB)p + 1}
\end{aligned}
$$

Butterworth Filter Transfer Functions

The Butterworth Approximation

The general form of a nth-order low-pass filter transfer function having no zeros can be written

$$T_{LP}(s) = K\frac{1}{1 + c_1(s/\omega_c) + c_2(s/\omega_c)^2 + \cdots + c_n(s/\omega_c)^n}$$

where K is the dc gain constant, ω_c is a normalization frequency, and the c_i are positive constants. The magnitude squared function is obtained by setting $s = j\omega$ and solving for $|T_{LP}(j\omega)|^2$. This function contains only even powers of ω and is of the form

$$|T_{LP}(j\omega)|^2 = K^2\frac{1}{1 + C_1(\omega/\omega_c)^2 + C_2(\omega/\omega_c)^4 + \cdots + C_n(\omega/\omega_c)^{2n}}$$

where the C_i are positive constants which are related to the c_i.

For the Butterworth filter, the constants C_i are chosen so that $|T_{LP}(j\omega)|^2$ approximates the magnitude squared function of an ideal low-pass filter in the maximally flat sense. The magnitude squared function for the ideal filter is defined by

$$
\begin{aligned}
|T_{LP}(j\omega)|^2 &= K^2 \text{ for } \omega \le \omega_c \\
&= 0 \text{ for } \omega > \omega_c
\end{aligned}
$$

To obtain the maximally-flat approximation, the C_i are chosen to make as many derivatives as possible of $|T_{LP}(j\omega)|^2$ equal to zero at $\omega = 0$. If the derivative of a function is zero, the derivative of the reciprocal of the function is also zero. It follows that the maximally flat condition can be imposed by solving for the constants C_i which make as many derivatives as possible of $\left[|T_{LP}(j\omega)|^2\right]^{-1}$ equal to zero at $\omega = 0$. Because the denominator polynomial of $|T_{LP}(j\omega)|^2$ is an even function, all odd-order derivatives are already zero at $\omega = 0$. For the second derivative to be zero, we must have $C_1 = 0$. For the fourth derivative to be zero, we must have $C_2 = 0$. This procedure is repeated to obtain $C_i = 0$ for all i. However, we cannot set $C_n = 0$ because this would make the approximating function independent of frequency. Therefore, we set $C_i = 0$ for all $1 \le i \le n-1$ to obtain

$$|T_{LP}(j\omega)|^2 = K^2 \frac{1}{1 + C_n (\omega/\omega_c)^{2n}}$$

The first $2n - 1$ derivatives of this function are zero at $\omega = 0$.

It is standard to choose C_n to make $|T_{LP}(j\omega_c)|^2 = K^2/2$. This forces the $-3\,\mathrm{dB}$ frequency ω_3 to be equal to the normalization frequency ω_c. This condition requires $C_n = 1$. Thus the magnitude squared function of the nth-order Butterworth low-pass filter becomes

$$|T_{LP}(j\omega)|^2 = K^2 \frac{1}{1 + (\omega/\omega_c)^{2n}}$$

Figure 6 shows example plots of $|T_{LP}(j\omega)|$ for $1 \le n \le 5$ for the Butterworth low-pass filter. The plots assume that $K = 1$. The horizontal axis is the normalized radian frequency $v = \omega/\omega_c$. Each function has the value 1 at $v = 0$, the value 0.5 at $v = 1$, and approaches 0 as $v \to \infty$. As the order n increases, the width of the flat region in the passband is extended and the filter exhibits a sharper cutoff. The response characteristic is called maximally flat because there are no ripples in the passband response.
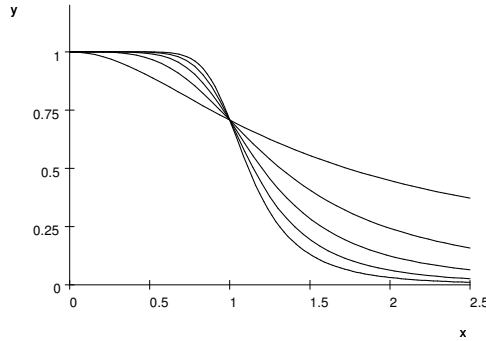

Figure 6   Plots of the Butterworth magnitude response for $1 \le n \le 5$.

To illustrate how the maximally flat condition is applied to a specific filter transfer function, consider the third-order low-pass function

$$T_{LP}(s) = K \frac{1}{1 + c_1 (s/\omega_c) + c_2 (s/\omega_c)^2 + c_3 (s/\omega_c)^3}$$

The magnitude squared function is given by

$$|T_{LP}(j\omega)|^2 = K^2 \frac{1}{1 + (c_1^2 - 2c_2)(\omega/\omega_c)^2 + (c_2^2 - 2c_1 c_3)(\omega/\omega_c)^4 + c_3^2 (\omega/\omega_c)^6}$$

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For this to be maximally flat with a $-3\,\text{dB}$ cutoff frequency of ω_c, we must have

$$
\begin{aligned}
c_1^2 - 2c_2 &= 0 \\
c_2^2 - 2c_1 c_3 &= 0 \\
c_3 &= 1
\end{aligned}
$$

Solution for c_1 and c_2 yields $c_1 = c_2 = 2$. Thus the Butterworth third-order low-pass transfer function is

$$
\begin{aligned}
T_{LP}(s) &= K \frac{1}{1 + 2\,(s/\omega_c) + 2\,(s/\omega_c)^2 + (s/\omega_c)^3} \\
&= K \frac{1}{1 + (s/\omega_c)} \times \frac{1}{1 + (s/\omega_c) + (s/\omega_c)^2}
\end{aligned}
$$

The maximally flat filters are called Butterworth filters after S. Butterworth who described the procedure for deriving the transfer functions in his 1930 paper "On the Theory of Filter Amplifiers" which was published in *Wireless Engineer*. The resulting denominator polynomials for $T_{LP}(s)$ are called Butterworth polynomials. The first six Butterworth polynomials in factored form are

$$
\begin{aligned}
b_1(x) &= (x+1) \\
b_2(x) &= \left(x^2 + 1.4142x + 1\right) \\
b_3(x) &= (x+1)\left(x^2 + x + 1\right) \\
b_4(x) &= \left(x^2 + 0.7654x + 1\right)\left(x^2 + 1.8478x + 1\right) \\
b_5(x) &= (x+1)\left(x^2 + 0.6180x + 1\right)\left(x^2 + 1.6180x + 1\right) \\
b_6(x) &= \left(x^2 + 0.5176x + 1\right)\left(x^2 + 1.4142x + 1\right)\left(x^2 + 1.9319x + 1\right)
\end{aligned}
$$

Even-Order Butterworth Filters

For an even-order Butterworth low-pass filter of order n, the transfer function can be written in the product form

$$
T_{LP}(s) = \prod_{i-1}^{n/2} \frac{1}{(s/\omega_c)^2 + (1/b_i)\,(s/\omega_c) + 1}
$$

The constants b_i are given by

$$
b_i = \frac{1}{2\sin\theta_i}
$$

where the θ_i are given by

$$
\theta_i = \frac{2i-1}{n} \times 90^\circ \text{ for } 1 \le i \le \frac{n}{2}
$$

Example 1 *Solve for the transfer functions of the second-order Butterworth low-pass and high-pass filters.*

Solution. For $n = 2$, there is only one second-order transfer function. The calculations are summarized as follows:

i	θ_i	b_i
1	45°	$1/\sqrt{2}$

The low-pass transfer function is given by

$$T(s) = K \frac{1}{(s/\omega_c)^2 + \sqrt{2}(s/\omega_c) + 1}$$

The high-pass transfer function is obtained by replacing s/ω_c with ω_c/s to obtain

$$
\begin{aligned}
T(s) &= K \frac{1}{(\omega_c/s)^2 + \sqrt{2}(\omega_c/s) + 1} \\
&= K \frac{(s/\omega_c)^2}{(s/\omega_c)^2 + \sqrt{2}(s/\omega_c) + 1}
\end{aligned}
$$

Odd-Order Butterworth Filters

For an odd-order Butterworth low-pass filter of order n, the transfer function can be written in the product form

$$T_{LP}(s) = \frac{1}{s/\omega_c + 1} \times \prod_{i=1}^{(n-1)/2} \frac{1}{(s/\omega_c)^2 + (1/b_i)(s/\omega_c) + 1}$$

The constants b_i are given by

$$b_i = \frac{1}{2\sin\theta_i}$$

where the θ_i are given by

$$\theta_i = \frac{2i-1}{n} \times 90° \text{ for } 1 \le i \le \frac{n-1}{2}$$

Example 2 *Solve for the transfer functions of the third-order Butterworth low-pass and high-pass filters.*

Solution. For $n = 3$, each transfer function contains one first-order polynomial and one second-order polynomial. The calculations for the second-order polynomial are summarized as follows:

i	θ_i	b_i
1	30°	1

The low-pass transfer function is given by

$$T_{LP}(s) = \frac{1}{(s/\omega_c) + 1} \times \frac{1}{(s/\omega_c)^2 + (s/\omega_c) + 1}$$

The high-pass transfer function is obtained by replacing s/ω_c with ω_c/s to obtain

$$
\begin{aligned}
T_{HP}(s) &= \frac{1}{(\omega_c/s) + 1} \times \frac{1}{(\omega_c/s)^2 + (\omega_c/s) + 1} \\
&= \frac{(s/\omega_c)}{(s/\omega_c) + 1} \times \frac{(s/\omega_c)^2}{(s/\omega_c)^2 + (s/\omega_c) + 1}
\end{aligned}
$$

The Cutoff Frequency

For the Butterworth low-pass and high-pass filter functions, the cutoff frequency ω_c is the frequency at which the magnitude-squared function is down by a factor of $1/2$. This is the $-3\,\mathrm{dB}$ frequency ω_3. For the band-pass and band-reject filter functions, the cutoff frequency ω_c is the so-called center frequency. There are two frequencies, one on each side of ω_c, at which the magnitude-squared function is down by a factor of $1/2$. These are the two $-3\,\mathrm{dB}$ frequencies. Let these be denoted by ω_{c1} and ω_{c2}. If these are specified, the center frequency ω_c and the parameter B in the frequency transformations are given by

$$\omega_c = \sqrt{\omega_{c1}\omega_{c2}} \qquad B = \frac{\omega_c}{\omega_{c2} - \omega_{c1}}$$

Chebyshev Filter Transfer Functions

The Chebyshev Approximation

In 1899, the Russian mathematician P. L. Chebyshev (also written Tschebyscheff, Tchebysheff, or Tchebicheff) described a set of polynomials $t_n(x)$ which have the feature that they ripple between the peak values of $+1$ and -1 for $-1 \leq x \leq +1$. His polynomials are widely used in filter approximations for frequencies that span the audio band to the microwave band. The first six Chebyshev polynomials are

$$
\begin{aligned}
t_1(x) &= x \\
t_2(x) &= 2x^2 - 1 \\
t_3(x) &= 4x^3 - 3x \\
t_4(x) &= 8x^4 - 8x^2 + 1 \\
t_5(x) &= 16x^5 - 20x^3 + 5x \\
t_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1
\end{aligned}
$$

Figure 7 shows the plots of the first four of these polynomials over the range $-2 \leq x \leq +2$.
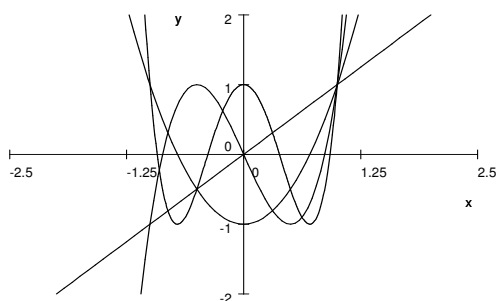

Figure 7   Plots of Chebyshev polynomials for $1 \leq n \leq 4$.

The Chebyshev approximation to the magnitude squared function of a low-pass filter is given by

$$|T_{LP}(j\omega)|^2 = K^2 \frac{1 + \epsilon^2 t_n^2(0)}{1 + \epsilon^2 t_n^2(\omega/\omega_c)}$$

where K is the dc gain constant and ϵ is a parameter which determines the amount of ripple in the approximation. For $\omega = 0$, it follows that $|T_{LP}(j\omega)|^2 = K^2$. For n odd, $t_n^2(0) = 0$ so that the numerator in $|T_{LP}(j\omega)|^2$ has the value 1. For $0 \leq \omega \leq \omega_c$, the denominator ripples

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between the values 1 and $1 + \epsilon^2$. This causes $|T_{LP}(j\omega)|^2$ to ripple between the values K^2 and $K^2/(1 + \epsilon^2)$. At $\omega = \omega_c$, it has the value $K^2/(1 + \epsilon^2)$. For $\omega > \omega_c$, $|T_{LP}(j\omega)|^2 \to 0$.

For n even, $t_n^2(0) = 1$ so that the numerator in $|T_{LP}(j\omega)|^2$ has the value $1 + \epsilon^2$. For $0 \leq \omega \leq \omega_c$, the denominator ripples between the values 1 and $1 + \epsilon^2$. This causes $|T_{LP}(j\omega)|^2$ to ripple between the values K^2 and $K^2(1 + \epsilon^2)$. At $\omega = \omega_c$, it has the value K^2. For $\omega > \omega_c$, $|T_{LP}(j\omega)|^2 \to 0$. The major difference between the even and odd order approximations is that the odd-order functions ripple down from the zero frequency value whereas the even-order functions ripple up.

Figure 8 shows example plots of $|T_{LP}(j\omega)|$ for the 0.5 dB ripple 4th and 5th order filters. The plots assume that $K = 1$. The horizontal axis is the normalized radian frequency $v = \omega/\omega_c$. The figure shows the 4th order approximation rippling up by 0.5 dB from its zero frequency value. The 5th order approximation ripples down by 0.5 dB from its zero frequency. Compared to the Butterworth filters, the Chebyshev filters exhibit a sharper cutoff at the expense of ripple in the passband. The more the ripple, the sharper the cutoff.
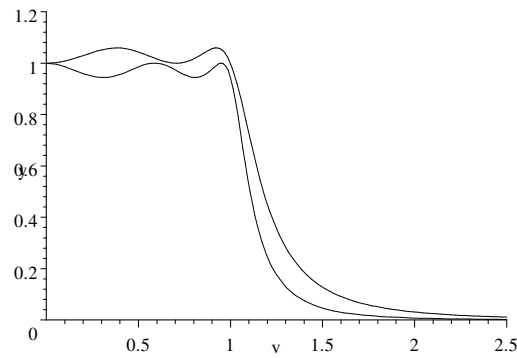

Figure 8   Plots of the magnitude responses of the 0.5 dB ripple 4th and 5th order Chebyshev filters.

The dB Ripple

The dB ripple for a Chebyshev filter is the peak-to-peak passband ripple in the Bode magnitude plot of the filter response. The dB ripple determines the parameter ϵ as follows:

$$\text{dB ripple} = 10\log\left(1 + \epsilon^2\right)$$

This can be solved for ϵ to obtain

$$\epsilon = \sqrt{10^{\text{dB}/10} - 1}$$

The Cutoff Frequency

Unlike the Butterworth filter, the cutoff frequency for the Chebyshev filter is not the frequency at which the response is down by 3 dB. The cutoff frequency is the frequency at which the Bode magnitude plot leaves the "equal-ripple box" in the filter passband. For the even-order low-pass filters, the gain at the cutoff frequency is equal to the zero frequency gain. For the odd-order low-pass filters, the gain at the cutoff frequency is down from the zero frequency gain by an amount equal to the dB ripple.

For the nth-order Chebyshev low-pass filter, the -3 dB frequency ω_3 can be related to the cutoff frequency ω_c by setting $|F(j\omega)|^2 = K^2/2$. The value of ω which satisfies this equation is the -3 dB frequency ω_3. If we let $x = \omega/\omega_c$, this leads to the equation

$$t_n(x) - \sqrt{\frac{1}{\epsilon^2} + 2t_n^2(0)} = 0$$

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The positive value of x which satisfies this equation is the desired solution. If the $-3\,\mathrm{dB}$ frequency for the low-pass and high-pass filters is specified, the required cutoff frequency ω_c is given by

$$\text{Low Pass (of order } n) \qquad \omega_c = \frac{\omega_3}{x}$$

$$\text{High Pass (of order } n) \qquad \omega_c = x\omega_3$$

For the band-pass and band-reject filter functions, the cutoff frequency ω_c is the so-called center frequency. There are two frequencies, one on each side of ω_c, at which the magnitude-squared function is down by a factor of $1/2$. These are the two $-3\,\mathrm{dB}$ frequencies. Let these be denoted by ω_{c1} and ω_{c2}. If these are specified, the center frequency ω_c and the parameter B in the frequency transformations are given by

$$\frac{\omega^2 - \omega_{c1}\omega_{c2}}{\omega\omega_c} = \frac{\omega_{c1} - \omega_{c2}}{\omega_c} = \frac{\omega_{c2} - \omega_{c1}}{\omega_c}$$

$$\text{Band Pass (of order } 2n) \qquad \omega_c = \sqrt{\omega_{c1}\omega_{c2}} \qquad B = \frac{x\omega_c}{\omega_{c2} - \omega_{c1}}$$

$$\text{Band Reject (of order } 2n) \qquad \omega_c = \sqrt{\omega_{c1}\omega_{c2}} \qquad B = \frac{\omega_c/x}{\omega_{c2} - \omega_{c1}}$$

where ω_3 is the $-3\,\mathrm{dB}$ radian frequency. For any x, there are two values of ω_3 which satisfy the band-pass and band-reject relations. Denote these by ω_a and ω_b. The geometric mean of these two frequencies must equal ω_c, i.e. $\omega_c = \sqrt{\omega_a\omega_b}$. In a design from specifications, the $-3\,\mathrm{dB}$ frequencies might be specified for the band-pass and band-reject filters. In this case, the required values for the center frequency and the parameter B in the frequency transformation can be solved for.

Example 3 *An 4th-order* $0.5\,\mathrm{dB}$ *ripple low-pass filter is to have the* $-3\,\mathrm{dB}$ *cutoff frequency* $f_3 = 10\,\mathrm{kHz}$. *Calculate the required cutoff frequency* f_c.

Solution. The parameter ϵ has the value $\epsilon = \sqrt{10^{0.5/10} - 1} = 0.34931$. We must solve for the positive real root of the equation

$$8x^4 - 8x^2 + 1 - \sqrt{\frac{1}{0.34931^2} + 2} = 0$$

The desired root is $x = 1.1063$. It follows that the required cutoff frequency is $f_c = f_3/x = 9.039\,\mathrm{kHz}$.

Example 4 *An 4th-order* $0.5\,\mathrm{dB}$ *ripple high-pass filter is to have the* $-3\,\mathrm{dB}$ *cutoff frequency* $f_3 = 10\,\mathrm{kHz}$. *Calculate the required cutoff frequency* f_c.

Solution. The value of x from the low-pass example gives $f_c = xf_3 = 11.063\,\mathrm{kHz}$.

Example 5 *An 8th-order* $0.5\,\mathrm{dB}$ *ripple band-pass filter is to have a center frequency* $f_c = 10\,\mathrm{kHz}$ *and a bandwidth* $f_{c2} - f_{c1} = 2\,\mathrm{kHz}$. *Note that the transfer function is obtained from that for the 4th-order LPF so that the band-pass filter is 8th order. Calculate the value of B in the low-pass to band-pass transformation.*

Solution. The value of x from the low-pass example gives $B = xf_c/(f_{c2} - f_{c1}) = 5.532$.

Example 6 *An 8th-order* $0.5\,\mathrm{dB}$ *ripple band-reject filter is to have a center frequency* $f_c = 10\,\mathrm{kHz}$ *and a bandwidth* $f_{c2} - f_{c1} = 2\,\mathrm{kHz}$. *Note that the transfer function is obtained from that for the 4th-order LPF so that the band-reject filter is 8th order.. Calculate the value of B in the low-pass to band-pass transformation.*

Solution. The value of x from the low-pass example gives $B = (f_c x^{-1})/(f_{c2} - f_{c1}) = 4.52$.

The Parameter h

To write the Chebyshev transfer functions, the parameter h is required. Given the order n and the ripple parameter ϵ, h is defined by

$$h = \tanh\left(\frac{1}{n}\sinh^{-1}\frac{1}{\epsilon}\right)$$

Even-Order Chebyshev Filters

For an even-order Chebyshev low-pass filter of order n, the transfer function can be written in the product form

$$T_{LP}(s) = \prod_{i=1}^{n/2}\frac{1}{(s/a_i\omega_c)^2 + (1/b_i)(s/a_i\omega_c) + 1}$$

The constants a_i and b_i are given by

$$a_i = \left(\frac{1}{1-h^2} - \sin^2\theta_i\right)^{1/2}\quad\text{for }1\le i\le n/2$$

$$b_i = \frac{1}{2}\left(1 + \frac{1}{h^2\tan^2\theta_i}\right)^{1/2}\quad\text{for }1\le i\le n/2$$

where the θ_i are given by

$$\theta_i = \frac{2i-1}{n}\times 90^\circ\quad\text{for }1\le i\le n/2$$

Example 7 *Solve for the transfer function of the $0.5\,\mathrm{dB}$ ripple fourth-order Chebyshev low-pass and high-pass filters.*

Solution. The parameters ϵ and h are given by

$$\epsilon = \left(10^{0.5/10}-1\right)^{1/2} = 0.3493$$

$$h = \tanh\left(\frac{1}{4}\sinh^{-1}\frac{1}{0.3493}\right) = 0.4166$$

For $n = 4$, there are two second-order polynomials in each transfer function. The calculations are summarized as follows:

i	θ_i	a_i	b_i
1	22.5°	1.0313	2.9406
2	67.5°	0.59703	0.7051

The low-pass transfer function is given by

$$\begin{aligned}T_{LP}(s) &= \frac{1}{(s/1.0313\omega_c)^2 + (1/2.9406)(s/1.0313\omega_c) + 1}\\ &\times\frac{1}{(s/0.59703\omega_c)^2 + (1/0.7051)(s/0.59703\omega_c) + 1}\end{aligned}$$

The high-pass transfer function is obtained by replacing s/ω_c with ω_c/s to obtain

$$
\begin{aligned}
T_{HP}(s) &= \frac{1}{(\omega_c/1.0313s)^2 + (1/2.9406)(\omega_c/1.0313s) + 1} \\
&\times \frac{1}{(\omega_c/0.59703s)^2 + (1/0.7051)(\omega_c/0.59703s) + 1} \\
&= \frac{(1.0313s/\omega_c)^2}{(1.0313s/\omega_c)^2 + (1/2.9406)(1.0313s/\omega_c) + 1} \\
&\times \frac{(0.59703s/\omega_c)^2}{(0.59703s/\omega_c)^2 + (1/0.7051)(0.59703s/\omega_c) + 1}
\end{aligned}
$$

Odd-Order Chebyshev Filters

For an odd-order Chebyshev low-pass filter of order n, the transfer function can be written in the form

$$
T_{LP}(s) = \frac{1}{s/a_{(n+1)/2}\omega_c + 1} \times \prod_{i=1}^{(n-1)/2} \frac{1}{(s/a_i\omega_c)^2 + (1/b_i)(s/a_i\omega_c) + 1}
$$

The constants a_i and b_i are given by

$$
a_{(n+1)/2} = \frac{h}{\sqrt{1-h^2}} \text{ for } i = (n+1)/2
$$

$$
a_i = \left(\frac{1}{1-h^2} - \sin^2\theta_i \right)^{1/2} \text{ for } 1 \le i \le (n-1)/2
$$

$$
b_i = \frac{1}{2}\left(1 + \frac{1}{h^2\tan^2\theta_i} \right)^{1/2} \text{ for } 1 \le i \le (n-1)/2
$$

where the θ_i are given by

$$
\theta_i = \frac{2i-1}{n} \times 90^\circ \text{ for } 1 \le i \le (n-1)/2
$$

Example 8 *Solve for the transfer function for the* $1\,\mathrm{dB}$ *ripple third-order Chebyshev low-pass and high-pass filters.*

Solution. The parameters ϵ and h are given by

$$
\epsilon = \left(10^{1/10} - 1 \right)^{1/2} = 0.5088
$$

$$
h = \tanh\left(\frac{1}{3}\sinh^{-1}\frac{1}{0.5088} \right) = 0.4430
$$

For $n = 3$, each transfer function contains one second-order polynomial and one first-order polynomial. The calculations are summarized as follows:

i	θ_i	a_i	b_i
1	30°	0.9971	2.0177
2	—	0.4942	—

The low-pass transfer function is given by

$$T_{LP}(s) = \frac{1}{(s/0.4942\omega_c) + 1} \times \frac{1}{(s/0.9971\omega_c)^2 + (1/2.0177)(s/0.9971\omega_c) + 1}$$

The high-pass function is obtained by replacing s/ω_c with ω_c/s to obtain

$$
\begin{aligned}
T_{HP}(s) &= \frac{1}{(\omega_c/0.4942s) + 1} \times \frac{1}{(\omega_c/0.9971s)^2 + (1/2.0177)(\omega_c/0.9971s) + 1} \\
&= \frac{(0.4942s/\omega_c)}{(0.4942s/\omega_c) + 1} \times \frac{(0.9971s/\omega_c)^2}{(0.9971s/\omega_c)^2 + (1/2.0177)(0.9971s/\omega_c) + 1}
\end{aligned}
$$

Elliptic Filter Transfer Functions

The elliptic filter is also known as the Cauer-Chebyshev filter. It is a filter which exhibits equal ripple in the passband and in the stop band. It can be designed to have a much steeper roll off than the Chebyshev filter. The approximation to the magnitude squared function of the elliptic low-pass filter is of the form

$$|T_{LP}(j\omega)|^2 = K^2 \frac{1 + \epsilon^2 R_n^2(0)}{1 + \epsilon^2 R_n^2(\omega/\omega_c)}$$

where K is the dc gain constant, ϵ determines the dB ripple, and $R_n(\omega/\omega_c)$ is the rational Chebychev function. This function has the form

$$
\begin{aligned}
R_n(x) &= \prod_{i=1}^{n/2} q_i^2 \frac{1 - (x/q_i)^2}{1 - (q_i x)^2} \qquad \text{for } n \text{ even} \\
&= x \prod_{i=1}^{(n-1)/2} q_i^2 \frac{1 - (x/q_i)^2}{1 - (q_i x)^2} \qquad \text{for } n \text{ odd}
\end{aligned}
$$

where $0 < q_i < 1$. From this definition, it follows that $R_n(x)$ satisfies the following properties:

$$R_n\left(\frac{1}{x}\right) = \frac{1}{R_n(x)} \qquad R_n(q_i) = 0 \qquad R_n\left(\frac{1}{q_i}\right) = \infty$$

$$
\begin{aligned}
R_n(0) &= \prod_{i=1}^{n/2} q_i^2 \qquad \text{for } n \text{ even} \\
&= 0 \qquad \text{for } n \text{ odd}
\end{aligned}
$$

$$
\begin{aligned}
R_n(1) &= (-1)^{n/2} \qquad \text{for } n \text{ even} \\
&= (-1)^{(n-1)/2} \qquad \text{for } n \text{ odd}
\end{aligned}
$$

$$
\begin{aligned}
R_n(\infty) &= \prod_{i=1}^{n/2} \frac{(-1)^i}{a_i^2} \qquad \text{for } n \text{ even} \\
&= \infty \qquad \text{for } n \text{ odd}
\end{aligned}
$$

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It is beyond the scope of this treatment to describe how the q_i are specified. For n even, the transfer function can be put into the form

$$T_{LP}(s) = \prod_{i=1}^{n/2} \frac{(s/c_i\omega_c)^2 + 1}{(s/a_i\omega_c)^2 + (1/b_i)(s/a_i\omega_c) + 1}$$

For n odd, the form is

$$T_{LP}(s) = \frac{1}{s/a_{(n+1)/2}\omega_c + 1} \times \prod_{i=1}^{(n-1)/2} \frac{(s/c_i\omega_c)^2 + 1}{(s/a_i\omega_c)^2 + (1/b_i)(s/a_i\omega_c) + 1}$$

These differ from the forms for the Chebyshev filter by the presence of zeros on the $j\omega$ axis. That is, $T_{LP}(j\omega) = 0$ for $\omega = c_i\omega_c$.

Figure 9 shows the plot of $|T_{LP}(jv)|$ for a $4th$-order elliptic filter, where $v = \omega/\omega_c$ is the normalized radian frequency. The filter has a dB ripple of 0.5 dB for $v \leq 1$. Like the Chebyshev filters, the gain ripples up from its dc value for the even order filter. There are two zeros in the response, one at $v = 1.59$ and the other at $v = 3.48$. For $v > 1.5$, the gain ripples between 0 and 0.0153. As v becomes large, the gain approaches the value 0.0153 which is 36.3 dB down from the dc value. Odd order elliptic filters have the property that the gain ripples down from the dc value and approaches 0 as v becomes large.
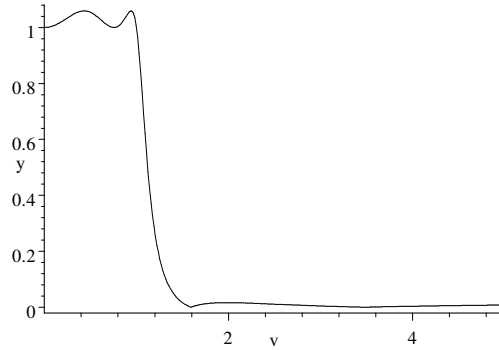

Figure 9   Magnitude response versus normalized frequency for the example elliptic filter.

The passband for the elliptic filter is the band defined by $\omega \leq \omega_c$. Like the Chebyshev filters, the dB ripple is defined in this band. The even-order filters ripple up from the dc gain, whereas the odd-order filters ripple down. At $\omega = \omega_c$, the gain of an even-order filter is equal to the dc gain, whereas the gain of an odd-order filter is down by an amount equal to the dB ripple. For $\omega > \omega_c$, the gain decreases rapidly and is equal to zero at the zero frequencies of the transfer function. Between adjacent zeros, the gain peaks up in an equal ripple fashion, i.e. with equal values at the peaks. Let the gain at the peaks between adjacent zeros have a dB level that is down from the dc gain by $A_{\min}$ dB. The stop band frequency ω_s is defined as the frequency between the cutoff frequency ω_c and the first zero frequency at which the gain is down by $A_{\min}$ dB. For $\omega > \omega_s$, the gain is down from the dc level by $A_{\min}$ dB or more.

The three parameters which define the alignment of an elliptic filter are the dB ripple in the passband, the ratio ω_s/ω_c of the stop band frequency to the cutoff frequency, and the minimum dB attenuation $A_{\min}$ in the stop band. The tables below give the values of the a_i, b_i, and c_i in the elliptic transfer functions for 0.5 dB ripple filters having ω_s/ω_c values of 1.5, 2.0, and 3.0 for orders $n = 2$ through $n = 5$.

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Elliptic Alignments for $\omega_s/\omega_c = 1.5$						Elliptic Alignments for $\omega_s/\omega_c = 2.0$					
n	$A_{\min}$	i	a_i	b_i	c_i	n	$A_{\min}$	i	a_i	b_i	c_i
2	8.3	1	1.2662	1.2275	1.9817	2	13.9	1	1.2617	1.0134	2.7321
3	21.9	1	1.0720	2.3672	1.6751	3	31.2	1	1.0717	1.9924	2.2701
		2	0.76695					2	0.69212		
4	36.3	1	1.0298	4.0390	1.5923	4	48.6	1	1.0308	3.4228	2.1432
		2	0.68690	0.74662	3.4784			2	0.64056	0.72416	4.9221
5	50.6	1	1.0158	6.2145	1.5574	5	66.1	1	1.0169	5.2811	2.0893
		2	0.75895	1.3310	2.3319			2	0.72457	1.2481	3.2508
		3	0.42597					3	0.39261		

Elliptic Alignments for $\omega_s/\omega_c = 3.0$					
n	$A_{\min}$	i	a_i	b_i	c_i
2	21.5	1	1.2471	0.91893	4.1815
3	42.8	1	1.0703	1.8159	3.4392
		2	0.65236		
4	64.1	1	1.0312	3.1267	3.2335
		2	0.61471	0.71264	7.6466
5	85.5	1	1.0174	4.8296	3.1457
		2	0.70455	1.2056	5.0077
		3	0.37452		

The Thompson Phase Approximation

Let $v_i(t)$ be the input voltage to a filter. If the output voltage is given by $v_o(t) = v_i(t - \tau)$, the filter is said to introduce a time delay of τ seconds to the signal. The transfer function of such a filter is given by

$$T(s) = \frac{V_o(s)}{V_i(s)} = e^{-\tau s}$$

For $s = j\omega$, the phase of the transfer function in radians is given by

$$\varphi(\omega) = -\omega\tau = -2\pi f\tau$$

Thus the phase lag through the filter is directly proportional to the frequency. A filter having such a phase function is called a linear-phase filter. The Thompson approximation is a low-pass filter which has a phase function that approximates a linear-phase function. It can only be realized as a low-pass filter. This is because a time delay cannot be realized with, for example, a high-pass filter. The Thompson filter is also called a Bessel filter. This is because Bessel polynomials are used in obtaining the filter coefficients.

Given the phase function $\varphi(\omega)$ for any filter, there are two delays which can be defined. These are the phase delay τ_φ and the group delay τ_g. These are given by

$$\tau_\varphi = -\frac{\varphi(\omega)}{\omega} = -\frac{\varphi(\omega)}{2\pi f}$$

$$\tau_g = -\frac{d\varphi(\omega)}{d\omega} = -\frac{1}{2\pi}\frac{d\varphi(\omega)}{df}$$

These are equal for a linear-phase filter.

Let n be the order of the filter. For n even, the transfer function can be put into the form

$$T_{LP}(s) = \prod_{i=1}^{n/2} \frac{1}{(s/a_i\omega_c)^2 + (1/b_i)(s/a_i\omega_c) + 1}$$

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For n odd, the form is

$$T_{LP}(s) = \frac{1}{s/a_{(n+1)/2}\omega_c + 1} \times \prod_{i=1}^{(n-1)/2} \frac{1}{(s/a_i\omega_c)^2 + (1/b_i)(s/a_i\omega_c) + 1}$$

where the time delay of the filter is given by $\tau = 1/\omega_c$. It is beyond the scope of the treatment here to describe how the a_i and b_i are obtained. The tables below give the values for $n = 1$ to $n = 6$. Also given is the ratio of the $-3\,\mathrm{dB}$ frequency f_3 to the cutoff frequency f_c.

Thompson Alignments for $1 \le n \le 4$

n	f_3/f_c	i	a_i	b_i
1	1	1	1.0	
2	1.3617	1	1.7321	0.57735
3	1.7557	1	2.5415	0.69105
		2	2.3222	
4	2.1139	1	3.0233	0.52193
		2	3.3894	0.80554

Thompson Alignments for $5 \le n \le 6$

n	f_3/f_c	i	a_i	b_i
5	2.4274	1	3.7779	0.56354
		2	4.2610	0.91648
		3	3.64674	
6	2.8516	1	4.3360	0.51032
		2	4.5665	0.61119
		3	5.1492	1.0233

Figure 10 shows the plot of $|T(j2\pi f)|$ as a function of frequency for the $4th$-order Thompson filter for the case $\omega_c = 1$ rad/s. This choice gives a delay of 1 second. The $-3\,\mathrm{dB}$ frequency of the filter is $f_3 = 2.1139\omega_0/2\pi = 0.3364\,\mathrm{Hz}$. If the $-3\,\mathrm{dB}$ frequency is multiplied by a constant, the delay time is divided by that constant. Fig. 11 shows the phase response of the filter. The phase delay is shown plotted in Fig. 12. It can be seen from this figure that the delay is approximately equal to 1 second up to the $-3\,\mathrm{dB}$ frequency of the filter.
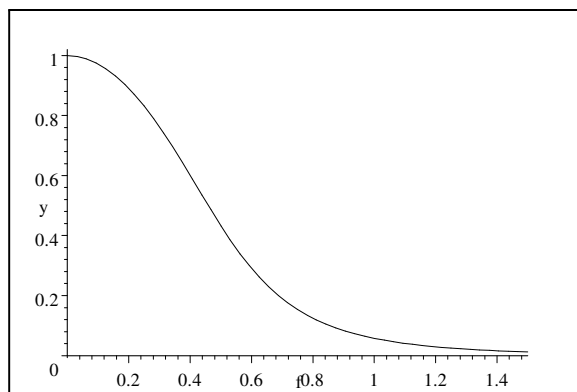

Figure 10   Plot of $|T(j2\pi f)|$ versus f.

Second-Order Filter Topologies

Sallen-Key Low-Pass Filter

Figure 13 shows the circuit diagram of the second-order Sallen-Key low-pass filter. The voltage-gain transfer function is given by

$$\frac{V_o}{V_i} = K\frac{1}{(s/\omega_0)^2 + (1/Q)(s/\omega_0) + 1}$$

where K is the dc gain constant, ω_0 is the radian resonance frequency, and Q is the quality factor. These are given by
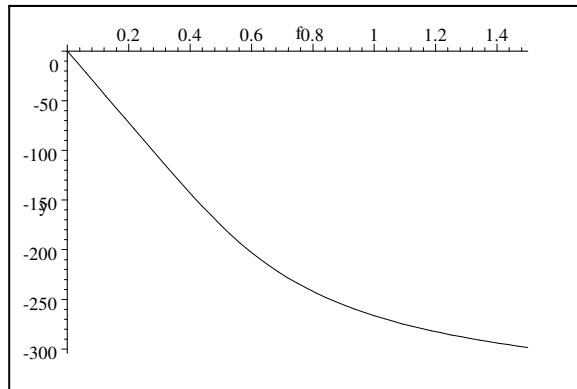
$$K = 1 + \frac{R_4}{R_3}$$

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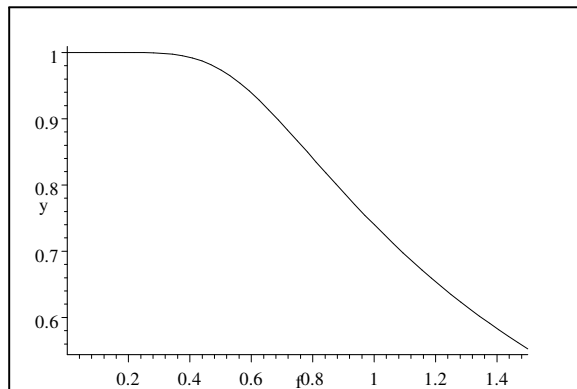Figure 11   Plot of the phase in degrees of $T\left(j2\pi f\right)$ versus f.

Figure 12   Plot of the phase delay in seconds of $T\left(j2\pi f\right)$ versus f.

$$\omega_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$

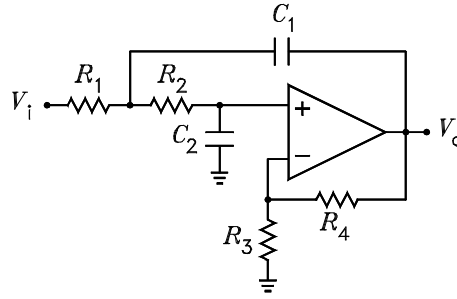$$Q = \frac{\sqrt{R_1 R_2 C_1 C_2}}{(1-K) R_1 C_1 + (R_1 + R_2) C_2}$$

Figure 13   Second-order Sallen-Key low-pass filter.

Special Case 1

Let $K = 1$ (R_4 a short and R_3 an open). If R_1, R_2, ω_0, and Q are specified, C_1 and C_2 are given by

$$C_1 = \frac{Q}{\omega_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

$$C_2 = \frac{1}{Q\omega_0 (R_1 + R_2)}$$

The filter is often realized with $R_1 = R_2$.

Special Case 2

Capacitor values are often difficult to obtain. It is possible to specify C_1 and C_2 and calculate the resistors. Let $K = 1$ (R_4 a short and R_3 an open). If C_1, C_2, ω_0, and Q are specified, R_1 and R_2 are given by

$$R_1, R_2 = \frac{1}{2Q\omega_0 C_2} \left[1 \pm \sqrt{1 - 4Q^2 \frac{C_2}{C_1}} \right]$$

Any value for the ratio C_2/C_1 can be chosen provided $4Q^2 C_2/C_1 \leq 1$. Note that the values for R_1 and R_2 are interchangeable.

Example 9 *Design a unity-gain second-order Sallen-Key low-pass filter with $f_0 = 1\,\text{kHz}$ and $Q = 1/\sqrt{2}$.*

   Solution. Let $C_1 = 0.1\,\mu\text{F}$ and $C_2 = 0.022\,\mu\text{F}$. R_1 and R_2 are given by

$$
\begin{aligned}
R_1, R_2 &= \frac{\sqrt{2}}{2 \times 2\pi 1000 \times 0.022 \times 10^{-6}} \left[1 \pm \sqrt{1 - 4 \times \frac{1}{2} \times 0.22} \right] \\
&= 1.29\,\text{k}\Omega \text{ and } 8.94\,\text{k}\Omega
\end{aligned}
$$

Either value may be assigned to R_1. The other value is then assigned to R_2.

Special Case 3

Let $R_1 = R_2 = R$ and $C_1 = C_2 = C$. If C, ω_0, and Q are specified, R and K are given by

$$R = \frac{1}{\omega_0 C}$$

$$K = 3 - \frac{1}{Q}$$

Infinite-Gain Multi-Feedback Low-Pass Filter

Figure 14 shows the circuit diagram of the second-order infinite-gain multi-feedback low-pass filter. The voltage-gain transfer function is given by

$$\frac{V_o}{V_i} = -K\frac{1}{(s/\omega_0)^2 + (1/Q)\,(s/\omega_0) + 1}$$

where K is the dc gain constant, ω_0 is the radian resonance frequency, and Q is the quality factor. These are given by

$$K = \frac{R_3}{R_1}$$

$$\omega_0 = \frac{1}{\sqrt{R_2 R_3 C_1 C_2}}$$

$$Q = \frac{\sqrt{R_2 R_3 C_1/C_2}}{R_2 + R_3 + R_2 R_3/R_1}$$
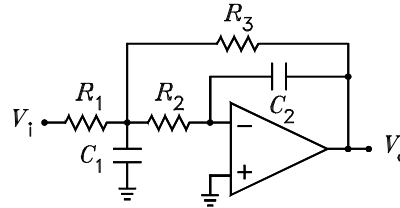
Figure 14   Second-order infinite gain multi-feedback low-pass filter.

Special Case 1

Let $R_1 = R_2 = R_3$ so that $K = -1$. If R, ω_0, and Q are specified, C_1 and C_2 are given by

$$C_1 = \frac{Q}{\omega_0} \left(\frac{2}{R_1} + \frac{1}{R_2} \right)$$

$$C_2 = \frac{1}{Q\omega_0\,(R_1 + 2R_2)}$$

The filter is often designed with $R_1 = R_2$.

Special Case 2

Capacitor values are often difficult to obtain. It is possible to specify C_1 and C_2 and calculate the resistors. Let $K = -1$ ($R_3 = R_1$). If C_1, C_2, ω_0, and Q are specified, $R_1/2$ and R_2 are given by

$$\frac{R_1}{2}, R_2 = \frac{1}{4Q\omega_0 C_2}\left[1 \pm \sqrt{1 - 8Q^2\frac{C_2}{C_1}}\right]$$

Any value for the ratio C_2/C_1 can be chosen provided $8Q^2 C_2/C_1 \leq 1$. Note that the values for $R_1/2$ and R_2 are interchangeable.

Example 10 *Design a unity-gain second-order infinite-gain multi-feedback low-pass filter with $f_0 = 1\,\text{kHz}$ and $Q = 1/\sqrt{2}$.*

Solution. Let $C_1 = 0.1\,\mu\text{F}$ and $C_2 = 0.01\,\mu\text{F}$. $R_1/2$ and R_2 are given by

$$
\begin{aligned}
\frac{R_1}{2}, R_2 &= \frac{\sqrt{2}}{4 \times 2\pi 1000 \times 0.01 \times 10^{-6}}\left[1 \pm \sqrt{1 - 8 \times \frac{1}{2} \times 0.1}\right] \\
&= 1.27\,\text{k}\Omega \text{ and } 9.99\,\text{k}\Omega
\end{aligned}
$$

The filter can be designed either with $R_1 = 2.54\,\text{k}\Omega$ and $R_2 = 9.99\,\text{k}\Omega$ or with $R_1 = 20\,\text{k}\Omega$ and $R_2 = 1.27\,\text{k}\Omega$.

Second-Order Low-Pass Filter -3dB Cutoff Frequency

The $-3\,\text{dB}$ radian cutoff frequency ω_3 of the filters in Figures 13 and 14 is given by

$$\omega_3 = \omega_0 \left[\left(1 - \frac{1}{2Q^2}\right) + \sqrt{\left(1 - \frac{1}{2Q^2}\right)^2 + 1}\right]^{1/2}$$

Sallen-Key High-Pass Filter

Figure 15 shows the circuit diagram for the second-order Sallen-Key high-pass filter. The voltage-gain transfer function is given by

$$\frac{V_o}{V_i} = K\frac{(s/\omega_0)^2}{(s/\omega_0)^2 + (1/Q)(s/\omega_0) + 1}$$

where K is the asymptotic high-frequency gain, ω_0 is the resonance frequency, and Q is the quality factor. These are given by

$$K = 1 + \frac{R_4}{R_3}$$

$$\omega_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$

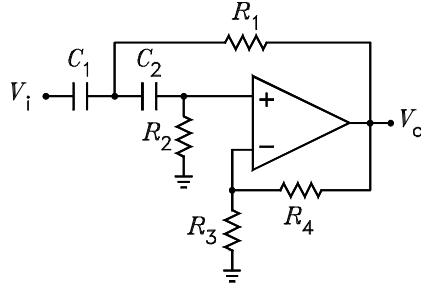$$Q = \frac{\sqrt{R_1 R_2 C_1 C_2}}{R_1(C_1 + C_2) + (1 - K)R_2 C_2}$$

Figure 15   Second-order Sallen-Key high-pass filter.

Special Case 1

Let $C_1 = C_2 = C$ and $K = 1$ (R_F a short and R_3 an open). If C, ω_0, and Q are specified, R_1 and R_2 are given by

$$R_1 = \frac{1}{2Q\omega_0 C}$$

$$R_2 = \frac{2Q}{\omega_0 C}$$

Special Case 2

Let $R_1 = R_2 = R$ and $C_1 = C_2 = C$. If R, ω_0, and Q are specified, C and K are given by

$$C = \frac{1}{\omega_0 R}$$

$$K = 3 - \frac{1}{Q}$$

Infinite-Gain Multi-Feedback High-Pass Filter

The second-order infinite-gain multi-feedback high-pass filter is not a stable circuit in practice. This is because its input node connects through two series capacitors to the V_- node of the op amp. At high frequencies, the input node becomes shorted to a virtual ground. This can cause oscillation problems in the source that drives the filter. Therefore, this topology is not recommended.

Second-Order High-Pass Filter -3dB Cutoff Frequency

The $-3\,\mathrm{dB}$ lower radian cutoff frequency ω_3 for the circuit in Figure 15 is given by

$$\omega_3 = \omega_0 \left[\left(1 - \frac{1}{2Q^2} \right) + \sqrt{\left(1 - \frac{1}{2Q^2} \right)^2 + 1} \right]^{-1/2}$$

Sallen-Key Band-Pass Filter

Figure 16 shows the circuit diagram of the second-order Sallen-Key band-pass filter. The voltage-gain transfer function is given by

$$\frac{V_o}{V_i} = K \frac{(1/Q)\,(s/\omega_0)}{(s/\omega_0)^2 + (1/Q)\,(s/\omega_0) + 1}$$

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where K is the gain at resonance, ω_0 is the resonance frequency, and Q is the quality factor. These are given by

$$K = \frac{R_2}{R_1 + R_2} \times \frac{K_0 R_3 C_2}{(R_1 \| R_2)(C_1 + C_2) + R_3 C_2 [1 - K_0 R_1 / (R_1 + R_2)]}$$

$$\omega_0 = \frac{1}{\sqrt{(R_1 \| R_2) R_3 C_1 C_2}}$$

$$Q = \frac{\sqrt{(R_1 \| R_2) R_3 C_1 C_2}}{(R_1 \| R_2)(C_1 + C_2) + R_3 C_2 [1 - K_0 R_1 / (R_1 + R_2)]}$$

where K_0 is the gain from the non-inverting input of the op amp to the output. This is given by

$$K_0 = 1 + \frac{R_5}{R_4}$$
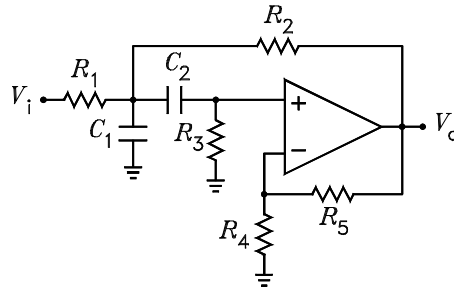
Figure 16   Second-order Sallen-Key band-pass filter.

Special Case

Let $R_1 = R_2 = R_3 = R$ and $C_1 = C_2 = C$. If Q, ω_0, and C are specified, R, K_0 and K are given by

$$R = \frac{\sqrt{2}}{\omega_0 C} \qquad K_0 = 4 - \frac{1}{Q^2} \qquad K = 4Q^2 - 1$$

Infinite-Gain Multi-Feedback Band-Pass Filter

Figure 17 shows the circuit diagram of the second-order infinite-gain multi-feedback band-pass filter. The voltage-gain transfer function is given by

$$\frac{V_o}{V_i} = -K \frac{(1/Q)(s/\omega_0)}{(s/\omega_0)^2 + (1/Q)(s/\omega_0) + 1}$$

where K is the gain at resonance, ω_0 is the resonance frequency, and Q is the quality factor. These are given by

$$K = \frac{R_3 C_1}{R_1 (C_1 + C_2)}$$

$$\omega_0 = \frac{1}{\sqrt{(R_1 \| R_2) R_3 C_1 C_2}}$$

$$Q = \frac{\sqrt{R_3 C_1 C_2 / (R_1 \| R_2)}}{C_1 + C_2}$$

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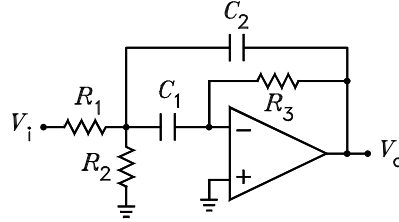Figure 17 Second-order infinite gain multi-feedback band-pass filter.

Special Case

Let $C_1 = C_2 = C$. For a specified K, ω_0, and Q, R_1, R_2, and R_3 are given by

$$R_1 = \frac{Q}{K\omega_0 C} \qquad R_2 = \frac{R_1}{(2Q^2/K) - 1} \qquad R_3 = \frac{2Q}{\omega_0 C}$$

Second-Order Band-Pass Filter Bandwidth

The $-3\,\mathrm{dB}$ bandwidth $\Delta\omega_3$ of a second-order band-pass filter is related to the resonance frequency ω_0 and the quality factor Q by

$$\Delta\omega_3 = \frac{\omega_0}{Q}$$

Let the lower and upper $-3\,\mathrm{dB}$ cutoff frequencies, respectively, be denoted by ω_a and ω_b. These frequencies are related to the bandwidth and the quality factor by

$$\Delta\omega_3 = \omega_b - \omega_a$$

$$\omega_0 = \sqrt{\omega_a \omega_b}$$

A Biquad Filter

A biquadratic transfer function, or biquad for short, is a transfer function that is the ratio of two second-order polynomials. It has the form

$$T_{\mathrm{BQ}}(s) = K\frac{(s/\omega_n)^2 + (1/Q_n)(s/\omega_n) + 1}{(s/\omega_d)^2 + (1/Q_d)(s/\omega_d) + 1}$$

where K, ω_n, Q_n, ω_d, and Q_d are constants. The function can be thought of as the sum of three transfer functions: a low-pass, a band-pass, and a high-pass. For the case $Q_n \to \infty$, the zeros are on the $j\omega$ axis and the transfer function exhibits a notch at $\omega = \omega_c$, i.e. $T_{\mathrm{BQ}}(j\omega_n) = 0$. Band-reject filters and elliptic filters have biquad terms of this form.

In general, the simplest biquad circuits to realize are the ones which require more than one op amp. Compared to the single op amp biquads, these circuits can be realized with only one capacitor value and they are usually easier to tweak. Fig. 18 shows the circuit diagram of an example three op amp biquad. For this circuit, we can write

$$V_a = -\left(R_4\|\frac{1}{Cs}\right)\left(\frac{V_i}{R_3} + \frac{V_b}{R}\right)$$

$$V_o = -R\left(\frac{V_a}{R} + \frac{V_i}{R_2}\right)$$

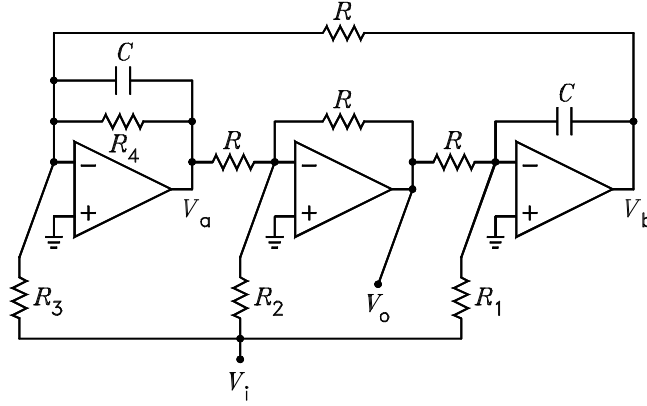$$V_b = \frac{-1}{Cs}\left(\frac{V_o}{R} + \frac{V_i}{R_1}\right)$$

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Figure 18   Biquad filter.

Although these equations look deceptively simple, a lot of algebra is required to solve for V_o/V_i. The solution is of the form of $T_{\text{BQ}}(s)$ given above. For a specified K, ω_d, Q_d, ω_n, Q_n, and C, the design equations for the circuit are

$$R = \frac{1}{\omega_d C}$$

$$R_1 = \frac{R}{K}$$

$$R_2 = \left(\frac{\omega_n}{\omega_d}\right)^2 R_1$$

$$R_3 = \frac{\omega_d R_2}{\omega_d/Q_d - \omega_n/Q_n}$$

$$R_4 = Q_d R$$

Example 11 *Design a third-order elliptic low-pass filter having* $0.5\,\text{dB}$ *ripple, a notch in its transfer function at* $f = 15{,}734\,\text{Hz}$*, a stop-band attenuation of* $21.9\,\text{dB}$*, and a stop band to cutoff frequency ratio of* $f_s/f_c = 1.5$*. The gain at dc is to be unity.*

Solution. From the $f_c/f_s = 1.5$ Elliptic filter table, the transfer function is given by

$$F(s) = \frac{1}{s/0.76695\omega_c + 1} \times \frac{(s/1.6751\omega_c)^2 + 1}{(s/1.0720\omega_c)^2 + (1/2.3672)(s/1.0720\omega_c) + 1}$$

where ω_c is the cutoff frequency. The transfer function has a null at $\omega = 1.6751\omega_c$. For the null to be at $15{,}734\,\text{Hz}$, we have $\omega_c = 2\pi 15734/1.6751 = 59{,}017$. Thus the transfer function is

$$F(s) = \frac{1}{s/45263 + 1} \times \frac{(s/98859)^2 + 1}{(s/63266)^2 + (1/2.367)(s/63266) + 1}$$

The biquadratic term can be realized with the three op amp biquad. Let $C = 1000\,\text{pF}$.

The element values are given by

$$R = \frac{1}{63266C} = 15.81\,\text{k}\Omega$$

$$R_1 = \frac{R}{1} = 15.81\,\text{k}\Omega$$

$$R_2 = \left(\frac{98859}{63266}\right)^2 R_1 = 38.60\,\text{k}\Omega$$

$$R_3 = \frac{63266 R_2}{63266/2.367 - 98859/\infty} = 2.367 R_2 = 91.37\,\text{k}\Omega$$

$$R_4 = 2.367 R = 37.42\,\text{k}\Omega$$

The first-order term can be realized as a passive filter at either the input or the output of the biquad. The preferred realization does not require a buffer stage for isolation. One way of accomplishing this is to divide R_1, R_2, and R_3 each into two series resistors, where the ratio of the resistors in each pair is the same. By voltage division, the voltage at the node where the resistors in each pair connect is the same for the three pairs. Thus these three nodes can be connected together. A capacitor to ground from this node then realizes the first-order term.

Let the ratio of the resistors in each pair be 0.33/0.67. Divide R_1 into series resistors of values $0.33 R_1 = 5.217\,\text{k}\Omega$ and $0.67 R_1 = 10.59\,\text{k}\Omega$. Similarly, divide R_2 into resistors of values $0.33 R_2 = 12.74\,\text{k}\Omega$ and $0.67 R_2 = 25.86\,\text{k}\Omega$ and divide R_3 into resistors of values $0.33 R_3 = 30.15\,\text{k}\Omega$ and $0.67 R_3 = 61.21\,\text{k}\Omega$. Let the smaller of the resistors in each pair connect to the V_i node. These three resistors are in parallel in the circuit and can be replaced with a single resistor of value $5.217\|12.74\|30.15 = 3.297\,\text{k}\Omega$. The time constant for the capacitor which sets the single-pole term in the transfer function is $R_p C$, where $R_p = 3.297\|10.59\|25.86\|61.21 = 2.209\,\text{k}\Omega$. Thus $C = 1/(45263 \times 2209) = 0.01\,\mu\text{F}$. The completed circuit is given in Fig. 19. The magnitude response is shown in Fig. 20.
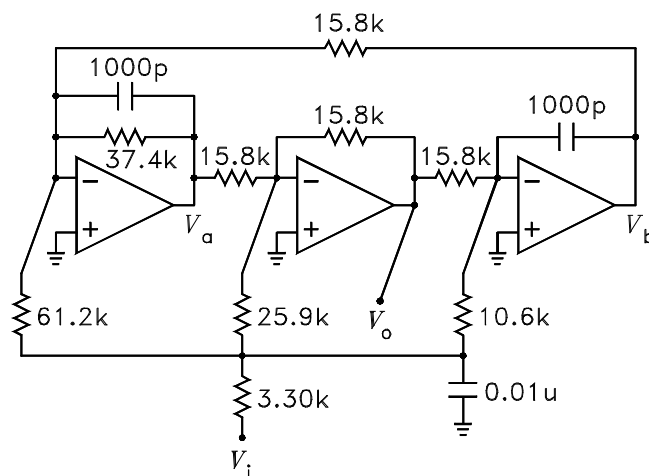

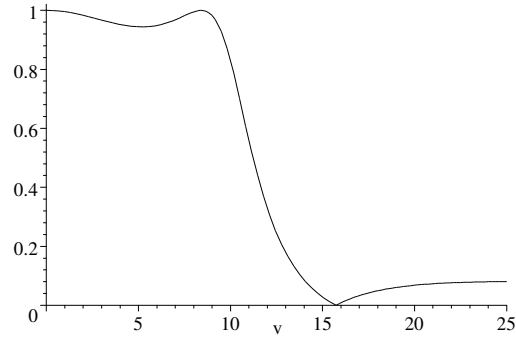Figure 19   Completed elliptic filter.

Figure 20   Magnitude response versus frequency in kHz.

A Second Biquad Filter

A second biquad filter that has its zeros on the $j\omega$ axis is shown in Fig. 21. The following equations can be written for the circuit:

$$V_a = -\frac{V_i}{R_1 C_1 s} - \frac{V_o}{R_3 C_1 s} \qquad V_b = -\frac{R_4}{R_5} V_i$$

$$V_c = \left(\frac{V_a}{R_2} + V_b C_2 s\right)\left(R_2 \| \frac{1}{C_2 s}\right) \qquad V_o = \left(1 + \frac{R_7}{R_6}\right) V_c$$

These equations can be solved for V_o/V_i to obtain

$$\frac{V_o}{V_i} = -K \frac{(s/\omega_n)^2 + 1}{(s/\omega_d)^2 + (1/Q_d)(s/\omega_d) + 1}$$

where

$$K = \frac{R_3}{R_1} \qquad \omega_n = \sqrt{\frac{R_5}{R_1 R_2 R_4 C_1 C_2}}$$

$$\omega_d = \sqrt{\frac{1 + R_7/R_6}{R_2 R_3 C_1 C_2}} \qquad Q_d = \sqrt{\left(1 + \frac{R_7}{R_6}\right)\frac{R_2 C_2}{R_3 C_1}}$$
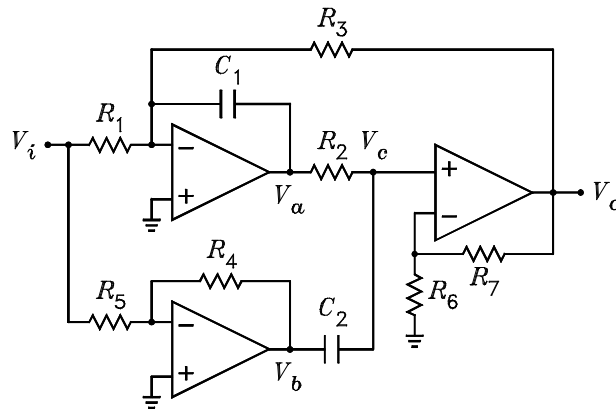

Figure 21   A second biquad circuit.

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Because there are more element values than equations that relate them, values must be assigned to some of the elements before the others can be calculated. Let values for C_1, C_2, K, R_5, and $1 + R_7/R_6$ be specified. The other element values are given by

$$R_1 = \frac{1 + R_7/R_6}{KQ_dC_1\omega_d} \qquad R_2 = \frac{Q_d}{\omega_dC_2} \qquad R_3 = KR_1 \qquad R_4 = K\left(\frac{\omega_d}{\omega_n}\right)^2 \frac{R_5}{1 + R_7/R_6}$$

Third-Order Sallen-Key Filter Circuits

The filter circuits given in this section consist of a first-order stage in cascade with a second-order stage with no buffer amplifier isolating the two stages. The transfer functions are difficult to derive.
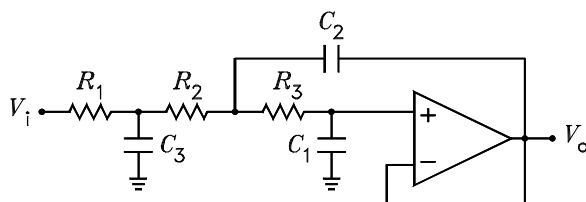
Low-Pass Filters

Figure 22 Third-order Sallen-Key low-pass filter.

Figure 22 shows the circuit diagram of a third-order unity-gain Sallen-Key low-pass filter. For $R_1 = R_2 = R_3 = R$, it can be shown that the transfer function for this filter is given by

$$\frac{V_o}{V_i} = \frac{1}{R^3C_1C_2C_3s^3 + 2R^2C_1(C_2 + C_3)s^2 + R(3C_1 + C_3)s + 1}$$

For a cutoff frequency ω_c, the element values for Butterworth and Chebyshev filters are given in the following table. The Butterworth filters are 3 dB down at $:\omega_c$ whereas the Chebyshev filters are down by an amount equal to the dB ripple.

Alignment	R_1	R_2	R_3	C_1	C_2	C_3
Butterworth	R	R	R	$0.20245/R\omega_c$	$3.5465/R\omega_c$	$1.3926/R\omega_c$
0.01 dB Chebyshev	R	R	R	$0.07130/R\omega_c$	$2.5031/R\omega_c$	$0.8404/R\omega_c$
0.03 dB Chebyshev	R	R	R	$0.07736/R\omega_c$	$3.3128/R\omega_c$	$1.0325/R\omega_c$
0.10 dB Chebyshev	R	R	R	$0.09691/R\omega_c$	$4.7921/R\omega_c$	$1.3145/R\omega_c$
0.30 dB Chebyshev	R	R	R	$0.08582/R\omega_c$	$7.4077/R\omega_c$	$1.6827/R\omega_c$
1.0 dB Chebyshev	R	R	R	$0.05872/R\omega_c$	$14.784/R\omega_c$	$2.3444/R\omega_c$

High-Pass Filters

Figure 23 shows the circuit diagram of a third-order unity-gain Sallen-Key high-pass filter. For $C_1 = C_2 = C_3 = C$, it can be shown that the transfer function for this filter is given by

$$\frac{V_o}{V_i} = \frac{R_1R_2R_3C^3s^3}{R_1R_2R_3C^3s^3 + R_2(R_1 + 3R_3)C^2s^2 + 2(R_2 + R_3)Cs + 1}$$

For a cutoff frequency ω_c, the element values for Butterworth and Chebyshev filters are given in the following table. The Butterworth filters are 3 dB down at $:\omega_c$ whereas the Chebyshev filters are down by an amount equal to the dB ripple.
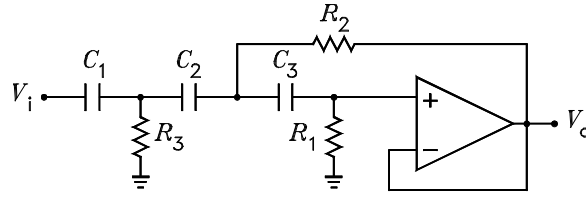
Figure 23   Third-order Sallen-Key high-pass filter.

Alignment	R_1	R_2	R_3	C_1	C_2	C_3
Butterworth	$4.93949/\omega_c C$	$0.28194/\omega_c C$	$0.71808/\omega_c C$	C	C	C
0.01 dB Chebyshev	$10.9130/\omega_c C$	$0.39450/\omega_c C$	$1.18991/\omega_c C$	C	C	C
0.03 dB Chebyshev	$10.09736/\omega_c C$	$0.30186/\omega_c C$	$0.96852/\omega_c C$	C	C	C
0.1 dB Chebyshev	$10.3188/\omega_c C$	$0.20868/\omega_c C$	$0.76075/\omega_c C$	C	C	C
0.3 dB Chebyshev	$11.65230/\omega_c C$	$0.13499/\omega_c C$	$0.59428/\omega_c C$	C	C	C
1.0 dB Chebyshev	$17.0299/\omega_c C$	$0.06764/\omega_c C$	$0.42655/\omega_c C$	C	C	C

Impedance Transfer Functions

RC Network

The impedance transfer function for a two-terminal RC network which contains only one capacitor and is not an open circuit at dc can be written

$$Z = R_{\text{dc}} \frac{1 + \tau_z s}{1 + \tau_p s}$$

where R_{dc} is the dc resistance of the network, τ_p is the pole time constant, and τ_z is the zero time constant. The pole time constant is the time constant of the network with the terminals open circuited. The zero time constant is the time constant of the network with the terminals short circuited. Figure 24(a) shows the circuit diagram of an example two-terminal RC network. The impedance transfer function can be written by inspection to obtain

$$Z = R_1 \frac{1 + R_2 C s}{1 + (R_1 + R_2) C s}$$

Figure 24   Example RC and RL impedance networks.

RL Network

The impedance transfer function for a two-terminal RL network which contains only one inductor and is not a short circuit at dc can be written

$$Z = R_{\text{dc}} \frac{1 + \tau_z s}{1 + \tau_p s}$$

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where R_{dc} is the dc resistance of the network, τ_p is the pole time constant, and τ_z is the zero time constant. The pole time constant is the time constant of the network with the terminals open circuited. The zero time constant is the time constant of the network with the terminals short circuited. Figure 24(b) shows the circuit diagram of an example two-terminal RL network. The impedance transfer function can be written by inspection to obtain

$$Z = R_1 \| R_2 \frac{1 + (L/R_2)\, s}{1 + [L/ (R_1 + R_2)]\, s}$$

Voltage Divider Transfer Functions

RC Network

The voltage-gain transfer function of a RC voltage-divider network containing only one capacitor and having a non-zero gain at dc can be written

$$\frac{V_o}{V_i} = k\frac{1 + \tau_z s}{1 + \tau_p s}$$

where k is the dc gain (C an open circuit), τ_p is the pole time constant, and τ_z is the zero time constant. The pole time constant is the time constant of the network with $V_i = 0$ and V_o open circuited. The zero time constant is the time constant of the network with $V_o = 0$ and V_i open circuited. Figure 25(a) shows the circuit diagram of an example RC network. The voltage-gain transfer function can be written by inspection to obtain

$$\frac{V_o}{V_i} = \frac{R_2 + R_3}{R_1 + R_2 + R_3} \times \frac{1 + (R_2 \| R_3)\, Cs}{1 + [(R_1 + R_2) \| R_3]\, Cs}$$

Figure 25(b) shows the circuit diagram of a second example RC network. The voltage-gain transfer function can be written by inspection to obtain

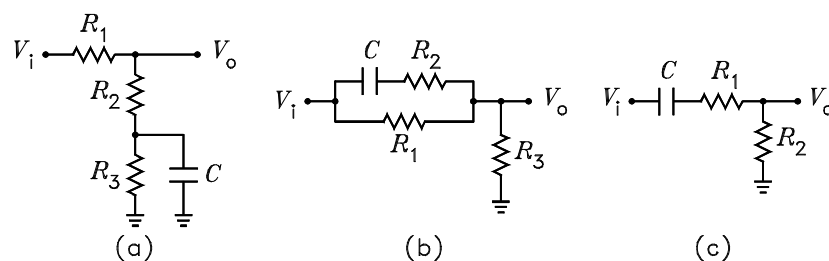$$\frac{V_o}{V_i} = \frac{R_3}{R_1 + R_3} \times \frac{1 + (R_1 + R_2)\, Cs}{1 + [(R_1 \| R_3) + R_2]\, Cs}$$

Figure 25   Example RC voltage divider networks.

High-Pass RC Network

The voltage-gain transfer function of a high-pass RC voltage-divider network containing only one capacitor can be written

$$\frac{V_o}{V_i} = k\frac{\tau_p s}{1 + \tau_p s}$$

where k is the infinite frequency gain (C a short circuit) and τ_p is the pole time constant. The pole time constant is calculated with $V_i = 0$ and V_o open circuited. Figure 25(c) shows

the circuit diagram of a third example RC network. The voltage-gain transfer function can be written by inspection to obtain

$$\frac{V_o}{V_i} = \frac{R_2}{R_1 + R_2} \times \frac{(R_1 + R_2)\,Cs}{1 + (R_1 + R_2)\,Cs}$$

RL Network

The voltage-gain transfer function of a RL voltage-divider network containing only one inductor and having a non-zero gain at dc can be written

$$\frac{V_o}{V_i} = k\frac{1 + \tau_z s}{1 + \tau_p s}$$

where k is the zero frequency gain (L a short circuit), τ_p is the pole time constant, and τ_z is the zero time constant. The pole time constant is the time constant of the network with $V_i = 0$ and V_o open circuited. The zero time constant is the time constant of the network with $V_o = 0$ and V_i open circuited. Figure 26(a) shows the circuit diagram of an example RL network. The voltage-gain transfer function can be written by inspection to obtain

$$\frac{V_o}{V_i} = \frac{R_2}{R_1 + R_2} \times \frac{1 + [L/(R_2\|R_3)]\,s}{1 + (L/[(R_1 + R_2)\,\|R_3])\,s}$$

Figure 26(b) shows the circuit diagram of a second example RL network. The voltage-gain transfer function can be written by inspection to obtain

$$\frac{V_o}{V_i} = \frac{R_3}{R_1 + R_3} \times \frac{1 + (L/R_1)\,s}{1 + (L/[R_1\|(R_2 + R_3)])\,s}$$
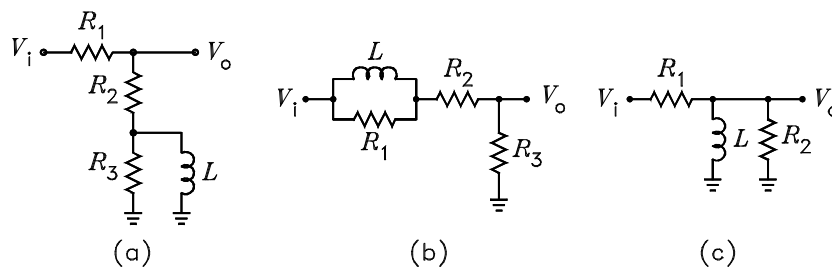

Figure 26   Example RL voltage divider circuits.

High-Pass RL Network

The voltage-gain transfer function of a high-pass RL voltage-divider network containing only one inductor can be written

$$\frac{V_o}{V_i} = k\frac{\tau_p s}{1 + \tau_p s}$$

where k is the infinite frequency gain (L an open circuit) and τ_p is the pole time constant. The pole time constant is calculated with $V_i = 0$ and V_o open circuited. Figure 26(c) shows the circuit diagram of a third example RL network. The voltage-gain transfer function can be written by inspection to obtain

$$\frac{V_o}{V_i} = \frac{R_2}{R_1 + R_2} \times \frac{[L/(R_1\|R_2)]\,s}{1 + [L/(R_1\|R_2)]\,s}$$