Controlled-Source Analogous Circuits and SPICE Models for Piezoelectric Transducers

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Abstract—**Transmission line analogous circuits for piezoelectric transducers are developed which employ controlled sources rather than the traditional transformer to model the coupling between the electrical and the mechanical systems. A novel method is used to derive each model that consists of adding a term that is equal to zero to one of the device electromechanical equations. When this is done, it is shown that the equations can be cast into the form of the familiar telegraphist's equations for the voltage and current on an electrical transmission line. The circuits are derived for both the thickness-mode piezoelectric transducer and the side-electrode bar piezoelectric transducer. SPICE models of the analogous circuits are presented and an example simulation is given.**

I. Introduction

Electroacoustic analogous circuits provide a powerful tool for the analysis of electro-mechano-acoustic systems. Most of the analogous circuits which have appeared in the literature employ transformers as the circuit elements that model the coupling between the electrical and mechanical systems and between the mechanical and the acoustical systems. The transformer apparently was chosen as a coupling element by early investigators because of the desire to obtain analogous circuits that could be assembled in the laboratory. An interesting historical discussion of this topic can be found in [1]. A disadvantage of the transformer is that it does not allow flexibility in the choice of the type of analogous circuits and it can lead to unrealizable impedance elements. For example, when the transformer is used to model the moving-coil electromagnetic-mechanical transducer commonly employed in loudspeakers and microphones, the mechanical analogous circuits must be mobility type analogs. In the case of Mason's model [2] of the piezoelectric thickness-mode transducer, the transformer results in a negative capacitor in the model.

In teaching classes, it has been my experience that students have no problems in writing circuit equations if the circuits contain controlled sources. In contrast, many students have difficulty with the analysis when a circuit contains a transformer. For this reason, I use controlled-source models in place of transformers in teaching courses. Not only is it straightforward to write equations for the circuits, but also the circuits can be easily analyzed with computer analysis programs such as SPICE. When controlled-source models are used for electroacoustic transducers, the analogous circuits are usually simpler than models that employ transformers. Not only can either impedance analogs or mobility analogs be used, but also a negative capacitor can be avoided when modeling piezoelectric transducers. In the latter case, the controlled-sources in the model may be frequency dependent. Although this might seem to complicate the models, writing equations for circuits that contain frequency-dependent controlled sources is no more difficult than writing equations for circuits that contain frequency dependent impedances, e.g. capacitors and inductors.

This paper presents the development of controlled-source analogous circuit models for elementary piezoelectric transducers. A novel technique is used to derive the models which consists of adding a term that is equal to zero to one of the device electromechanical equations. After the zero term is added, it is shown that the equations are in the form of the familiar telegraphist's equations for the voltage and current on an electrical transmission line. Both impedance analogous circuit models and mobility analogous circuit models are given for the thickness-mode transducer and the side-electrode bar transducer. In the impedance analogous circuits, voltage is analogous to force and current is analogous to velocity. In the mobility analogous circuits, voltage is analogous to velocity and current is analogous to force. SPICE implementations of two of the models are described.

Following the approach in [3], the functions representing network variables in this paper are assumed to be Laplace transforms of the original time functions. This provides a considerable simplification of the derivations without any loss in generality. It can be considered to be equivalent to the assumption that time variations are of the form *exp(st)*, where s is the complex frequency. Thus a multiplication by s in the Laplace transform domain represents the derivative operator in the time domain. A division by s in the Laplace transform domain represents the integration operator in the time domain. Each variable is a phasor, with an amplitude and a phase.

II. The Thickness-Mode Transducer

Fig. 1 shows the diagram of the thickness-mode piezoelectric transducer. Let the three dimensions of the crystal be denoted by ℓ_x, ℓ_y, and ℓ_z. A one-dimensional compressional wave is assumed that propagates in the z direction. In addition, it is assumed that the electric field intensity E and the electric flux density D are in the z direction. Let ζ be the particle displacement in the wave, u the particle velocity, and f the

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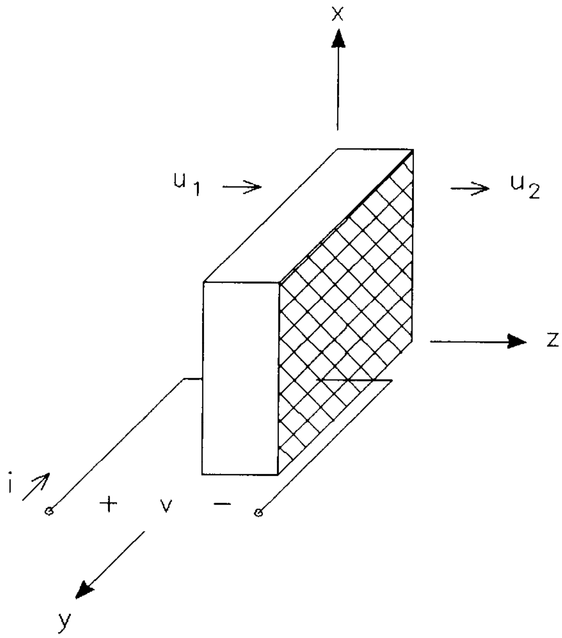

Fig. 1. Diagram of the thickness-mode transducer.

force. The equations which govern the wave are [2], [3]

$$\frac{df}{dz} = -\rho A_z s u \tag{1}$$

$$c\frac{d\zeta}{dz} = -\frac{1}{A_z}f + hD \tag{2}$$

$$E = -h\frac{d\zeta}{dz} + \frac{1}{\epsilon}D \tag{3}$$

where s is the complex frequency, ρ is the density, $A_z = \ell_x \ell_y$ is the crystal cross-sectional area that is perpendicular to the z axis, c is the relative elastic constant (N/m^2), h is the piezoelectric constant $(N \cdot m^4/C)$, and ϵ is the permittivity (F/m). These equations are valid in both piezoelectric crystals and piezoelectric polarized ceramics.

Let i be the current which flows in the external circuit that connects to the crystal electrodes. The charge q on the electrodes is related to the current by the equation $q = i/s$. Because the electric flux density is related to the charge by $D = q/A_z$, it follows that $D = i/(sA_z)$. The particle displacement is related to the particle velocity by $\zeta = u/s$. In (2), let $D = i/(sA_z)$ and $\zeta = u/s$. Because $dD/dz = 0$, it follows that $d(hi/s)/dz = 0$. Therefore, the term $d(hi/s)/dz$ can be subtracted from the left side of (1) without changing the equation. With these modifications, (1) and (2) can be rewritten

$$\frac{d}{dz}\left[f - \frac{h}{s}i\right] = -\rho A_z s u \tag{4}$$

$$\frac{du}{dz} = -\frac{s}{A_z c}\left[f - \frac{h}{s}i\right] \tag{5}$$

In (3), let $D = i/(sA_z)$ and $\zeta = u/s$. The voltage across the transducer electrodes is obtained by integrating the equation in z over the limits from $z = 0$ to $z = \ell_z$. It follows that the voltage is given by

$$v = \frac{h}{s}[u_1 - u_2] + \frac{1}{C_o s}i \tag{6}$$

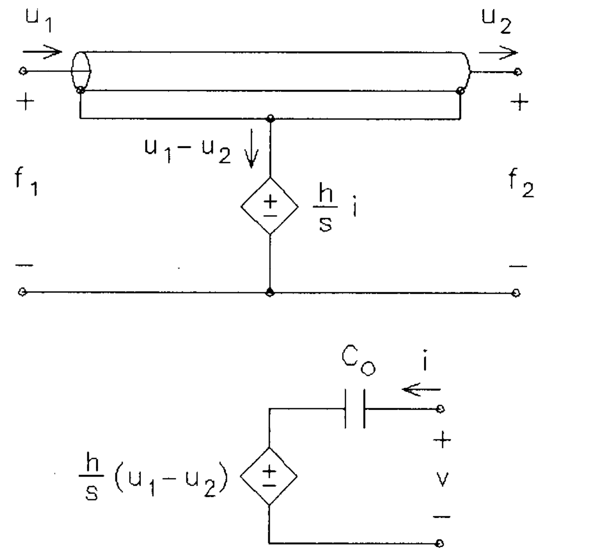

Fig. 2. Analogous circuit for the thickness-mode transducer in which the mechanical analog is an impedance type.

where $u_1 = u(0)$, $u_2 = u(\ell_z)$, and $C_o = \epsilon A_z/\ell_z$ is the capacitance between the electrodes.

The telegraphist's equations [4] for the voltage V and current I on an electrical transmission line are

$$\frac{dV}{dz} = -LsI \tag{7}$$

$$\frac{dI}{dz} = -CsV \tag{8}$$

where L is the inductance per unit length of the line and C is the capacitance per unit length. It can be seen that (4) and (5) are in the form of the telegraphist's equations for a transmission line on which the voltage V is analogous to the quantity $[f - (h/s)i]$ and the current I is analogous to u. The transmission line has a series inductance per unit length $L = \rho A_z$ and a shunt capacitance per unit length $C = 1/(A_z c)$. The phase velocity on the line is $u_p = 1/(LC)^{1/2} = (c/\rho)^{1/2}$ and the characteristic impedance is $Z_o = (L/C)^{1/2} = A_z(\rho c)^{1/2} = \rho A_z u_p$.

Analogous circuits which model (4), (5), and (6) are shown in Fig. 2. Like Redwood's version [3] of Mason's model [2], the circuit contains a transmission line. (The figure shows an unbalanced line for which the series inductance of the shield is zero. The circuit is also valid for a balanced line for which both leads exhibit series inductance.) Unlike those models, the circuit contains no transformer nor does it have a negative capacitor. The circuit consists of two parts, an electrical analogous circuit and a mechanical analogous circuit. The coupling between the circuits is modeled by two controlled sources. The controlled source in the mechanical part has the voltage hi/s. This voltage is a common-mode voltage at the two transmission line ports. Because the charge on the transducer electrodes is related to the current by $q = i/s$, the voltage is equal to hq. Thus the voltage source is controlled by the charge on the electrodes. The controlled source in the electrical part of the circuit has the voltage $h(u_1 - u_2)/s$. Because the particle displacement is related to the particle

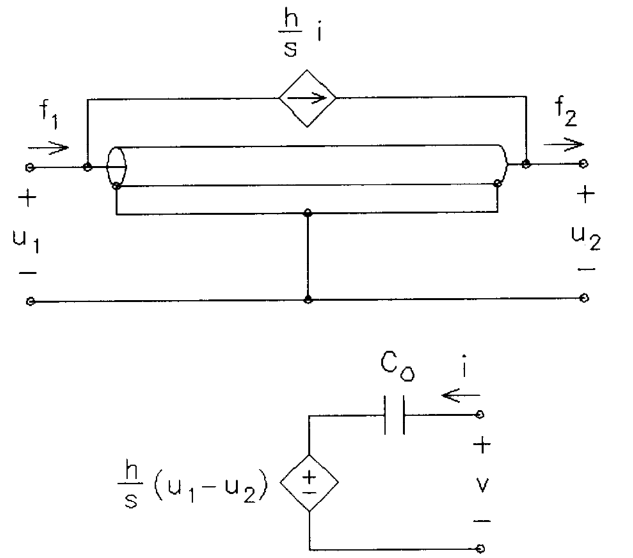

Fig. 3. Analogous circuit for the thickness-mode transducer in which the mechanical analog is a mobility type.

velocity by $\zeta = u/s$, this voltage is equal to $h(\zeta_1 - \zeta_2)$. Thus the voltage source is controlled by the difference between the particle displacements at two faces of the transducer.

A second analogous circuit can be formed if the voltage V on the transmission line is taken to be analogous to u and the current I is taken to be analogous to the quantity $[f-(h/s)i]$. The circuit is given in Fig. 3. The controlled source in the mechanical part of the circuit has the current hi/s. This current is a differential current between the two transmission line ports. The transmission line has a series inductance per unit length $L = 1/(A_z c)$ and a shunt capacitance per unit length $C = \rho A_z$. The phase velocity on the line is $u_p = 1/(LC)^{1/2} = (c/\rho)^{1/2}$ and the characteristic impedance is $Z_o = (L/C)^{1/2} = 1/(\rho A_z u_p)$. The mechanical part of the circuit in Fig. 3 is a mobility type analog in which voltage is analogous to particle velocity and current is analogous to force. In contrast, the mechanical part of the circuit in Fig. 2 is an impedance type analog in which voltage is analogous to force and current is analogous to particle velocity.

The $1/s$ term in the controlled source in the electrical parts of the circuits in Figs. 2 and 3 can be eliminated by making a Norton equivalent circuit of the series voltage source and capacitor. The Norton equivalent circuit consists of a current source of value $hC_o(u_1 - u_2)$ in parallel with a capacitor C_o. This transformation is shown in the SPICE models presented in the following.

III. The Side-Electrode Bar Transducer

Fig. 4 shows the diagram of the side-electrode bar transducer. This transducer can be used to generate or detect waves traveling in the z direction if the crystallographic X-axis is parallel to the x-axis in the figure. A one-dimensional compressional wave is assumed that propagates in the z direction. Equations (1) through (3) can be used to obtain an approximate plane wave solution that propagates in the "Young's modulus mode" [2], [3]. In this case, the electric

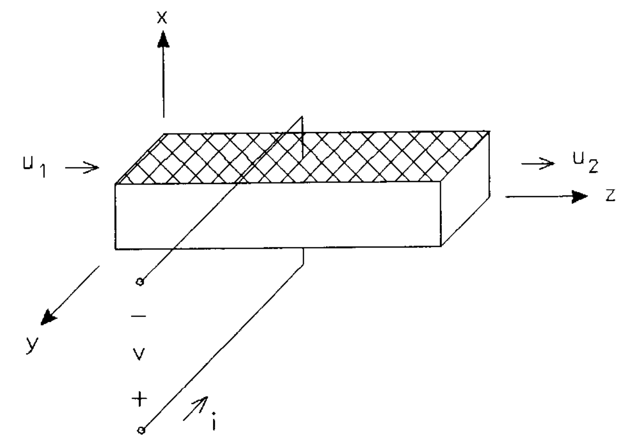

Fig. 4. Diagram of the side-electrode bar transducer.

field intensity E and the electric flux density D in (2) and (3) are in the x direction. The electric field in the crystal can be written $E = v/\ell_x = vC_o/(\epsilon A_x)$, where v is the voltage across the electrodes, $C_o = \epsilon A_x/\ell_x$ is the capacitance between the electrodes, and A_x is the cross-sectional area of the crystal that is perpendicular to the x-axis. The particle displacement can be written $\zeta = u/s$. When these substitutions are made in (2) and (3), the equations can be combined to obtain

$$\frac{du}{dz} = -\frac{s}{A_z(c-\epsilon h^2)}\left[f - \frac{hC_oA_z}{A_x}v\right] \tag{9}$$

Because $dE/dz = 0$, it follows that $d(hC_oA_zv/A_x)/dz = 0$. Therefore, this term can be subtracted from the left side of (1) without changing the equation. When this is done, (1) can be rewritten in the equivalent form

$$\frac{d}{dz}\left[f - \frac{hC_oA_z}{A_x}v\right] = -\rho A_z s u \tag{10}$$

The voltage across the transducer electrodes is given by (6).

Equations (9) and (10) are in the form of the telegraphist's equations for a transmission line on which the voltage V is analogous to the quantity $[f-(hC_oA_z/A_x)v]$ and the current I is analogous to u. The transmission line has a series inductance per unit length $L = \rho A_z$ and a shunt capacitance per unit length $C = 1/[A_z(c-\epsilon h^2)]$. The phase velocity on the line is $u_p = 1/(LC)^{1/2} = [(c-\epsilon h^2)/\rho]^{1/2}$ and the characteristic impedance is $Z_o = (L/C)^{1/2} = A_z[\rho(c-\epsilon h^2)]^{1/2} = \rho A_z u_p$. The analogous circuits which model the equations are shown in Fig. 5. The mechanical part of the circuit is an impedance type analog.

A second analogous circuit can be formed if the voltage V on the transmission line is taken to be analogous to u and the current I is taken to be analogous to the quantity $[f - (hC_oA_z/A_x)v]$. The circuit is given in Fig. 6. The mechanical part of the circuit is a mobility type analog. The transmission line has a series inductance per unit length $L = 1/[A_z(c-\epsilon h^2)]$ and a shunt capacitance per unit length $C = \rho A_z$. The phase velocity on the line is $u_p = 1/(LC)^{1/2} = [(c-\epsilon h^2)/\rho]^{1/2}$ and the characteristic impedance is $Z_o = (L/C)^{1/2} = 1/(\rho A_z u_p)$.

The circuits of Figs. 5 and 6 differ from those of Figs. 2 and 3 for the thickness-mode transducer only in the controlled

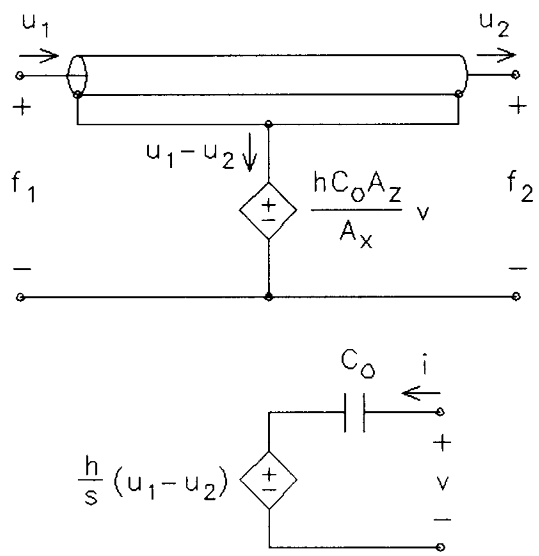

Fig. 5. Analogous circuit for the side-electrode bar transducer in which the mechanical analog is an impedance type.

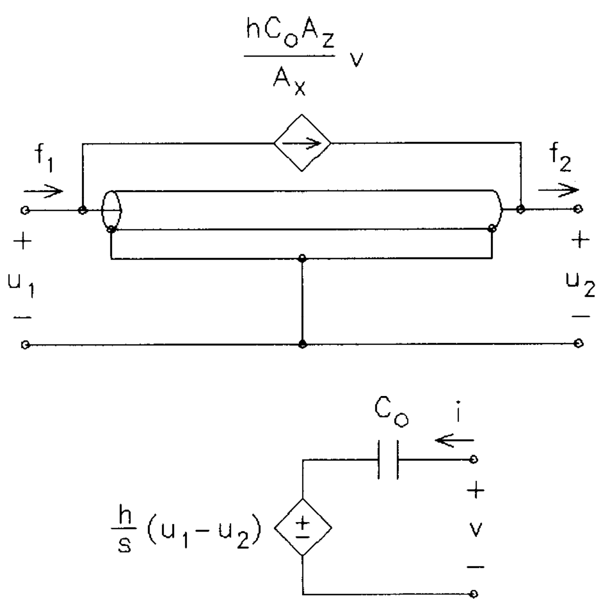

Fig. 6. Analogous circuit for the side-electrode bar transducer in which the mechanical analog is a mobility type.

source that is in the mechanical parts of the circuits. For the side-electrode bar transducer, the source is controlled by the voltage across the electrodes and is independent of frequency. For the thickness-mode transducer, the source is controlled by the current in the external leads and is inversely proportional to frequency, i.e., it is controlled by the charge on the electrodes.

As for the thickness-mode transducer, the $1/s$ term in the controlled source in the electrical parts of the circuits in Figs. 5 and 6 can be eliminated by making a Norton equivalent circuit of the series voltage source and capacitor. The Norton equivalent circuit consists of a current source of value $hC_o(u_1 - u_2)$ in parallel with a capacitor C_o. With this transformation, none of the sources in the models is frequency dependent. The Norton equivalent circuit is made in the SPICE model presented in the following.

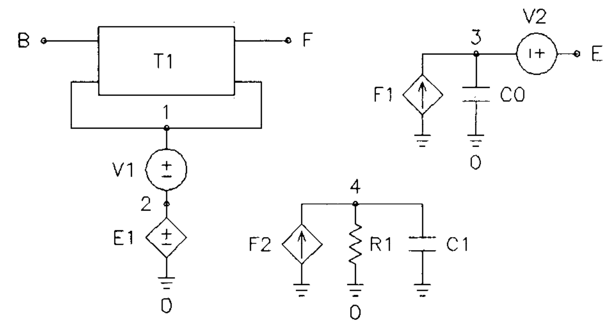

Fig. 7. First SPICE subcircuit for the thickness-mode transducer.

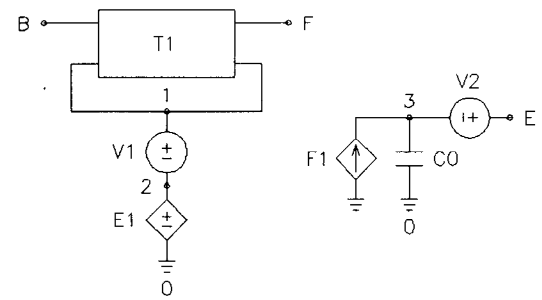

Fig. 8. Second SPICE subcircuit for the thickness-mode transducer and subcircuit for the side-electrode bar transducer.

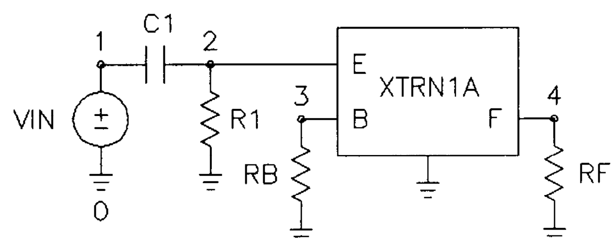

Fig. 9. SPICE circuit for the example simulation.

IV. SPICE Models

A comprehensive treatment of circuit analysis with SPICE is given in [6]. An introduction to the use of SPICE for the analysis of electroacoustic systems is given in [6]. A direct SPICE implementation of Mason's model of the piezoelectric transducer is given in [8], [9]. This section presents possible SPICE models of piezoelectric transducers that are based on the analogous circuits developed in this paper. Models are presented only for the circuits of Figs. 2 and 5 in which the mechanical analogous circuits are impedance type analogs. It is straightforward to form the models for the circuits of Figs. 3 and 6 in which the mechanical analogous circuits are mobility type analogs.

A SPICE implementation of the thickness-mode transducer model is shown in Fig. 7. The transmission line is represented by the block labeled T_1. Independent sources are represented by circular symbols. Controlled sources are represented by diamond shaped symbols. The independent voltage sources V_1 and V_2 are zero value sources which are used as ammeters in the circuit. To eliminate the need to realize a $1/s$ term in one of

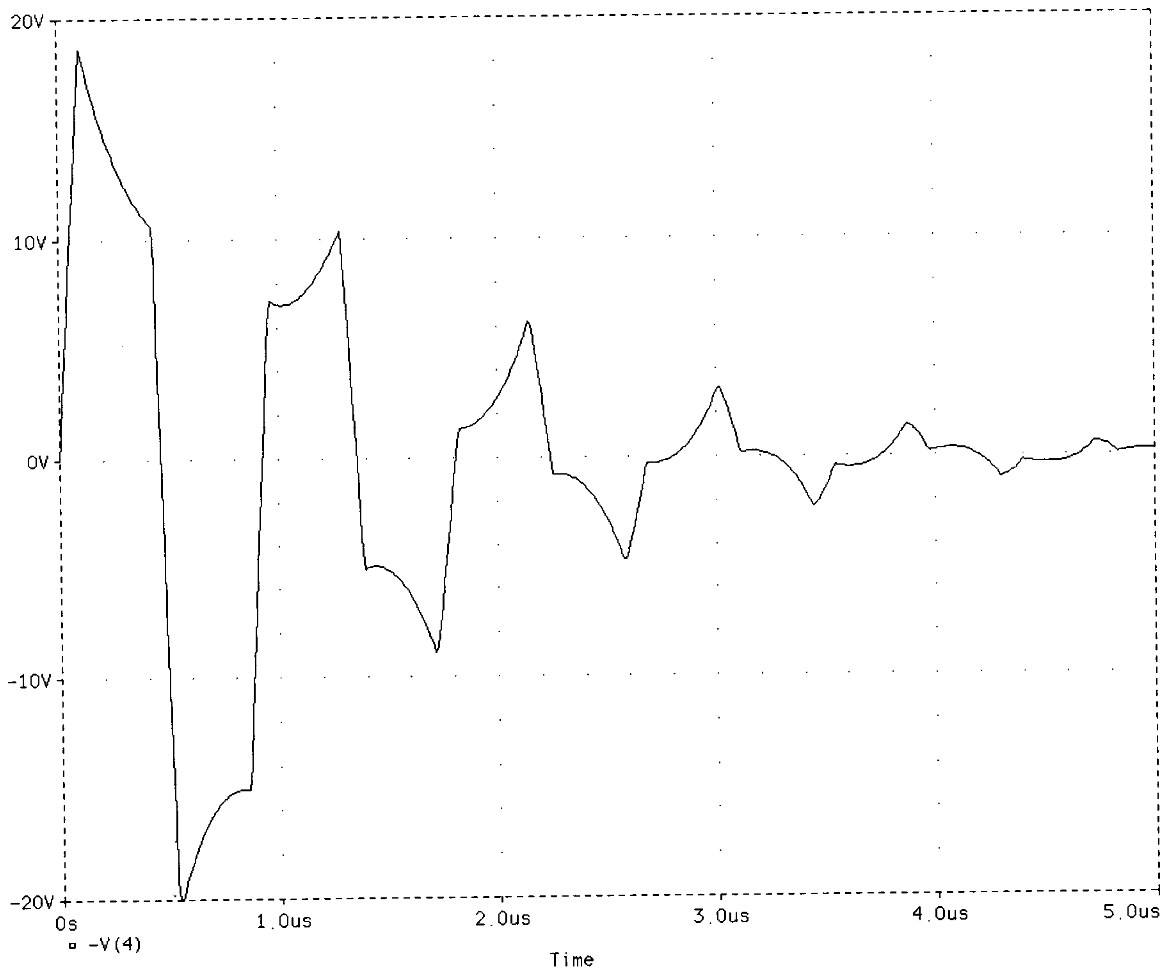

Fig. 10. Calculated transient response of the example transducer.

the controlled sources, a Norton equivalent circuit consisting of the dependent current source F_1 in parallel with the capacitor C_o is used to realize the electrical analogous circuit. The current F_1 is given by $F_1 = hC_o \times I(V_1)$, where $I(V_1)$ is the current through V_1. The voltage across the dependent voltage source E_1 is given by $E_1 = V(4)$, where $V(4)$ is the voltage at node 4, i.e. the voltage across C_1. The dependent current source F_2 which charges C_1 is given by $F_2 = h \times I(V_2)$, where $I(V_2)$ is the current through V_2. For $C_1 = 1\ F$, the dependent voltage E_1 is equal to h multiplied by the integral of the current through V_2. The integration operation realizes the $1/s$ term in the transfer function for E_1. Resistor R_1 is included to prevent node 4 from being a floating node. This resistor can be made large enough so that it is effectively an open circuit. (This is true if $1/(2\pi R_1 C_1)$ is much smaller than the lowest frequency of interest. For $R_1 = 1k\Omega$ and $C_1 = 1F$, this frequency is $1.6 \times 10^{-4}\ Hz$.) The SPICE subcircuit code for the model is given below. Input numerical values are indicated by letters in quotes. The node labels E, B, and F, respectively, denote the electrical, back, and front ports. The transmission line is defined by its characteristic impedance $Z_o = \rho A_z u_p$ and its time delay $\tau_d = \ell_z/u_p$, where $u_p = (c/\rho)^{1/2}$ is the phase velocity.

*THICKNESS-MODE TRANSDUCER SUBCIRCUIT A
.SUBCKT XTRN1A E B F
T1 B 1 F 1 Z0="Z_o" TD="τ_d"
V1 1 2
E1 2 0 4 0 1
V2 E 3
C0 3 0 "C_o"
F1 0 3 V1 "hC_o"
F2 0 4 V2 "h"
R1 4 0 1E3
C1 4 0 1
.ENDS

Fig. 8 shows an alternate implementation for the thickness-mode transducer that does not require a capacitor to realize an integration. This simpler model must be implemented with the analog behavioral modeling feature of *PSpice*. (*PSpice* for personal computers is a product of MicroSim Corp.) The code for the circuit is given below. The $1/s$ term in the gain expression for source E_1 is implemented by the LAPLACE function in *PSpice*.

*THICKNESS-MODE TRANSDUCER SUBCIRCUIT B
.SUBCKT XTRN1B E B F
T1 B 1 F 1 Z0 = "Z_o" TD = "τ_d"

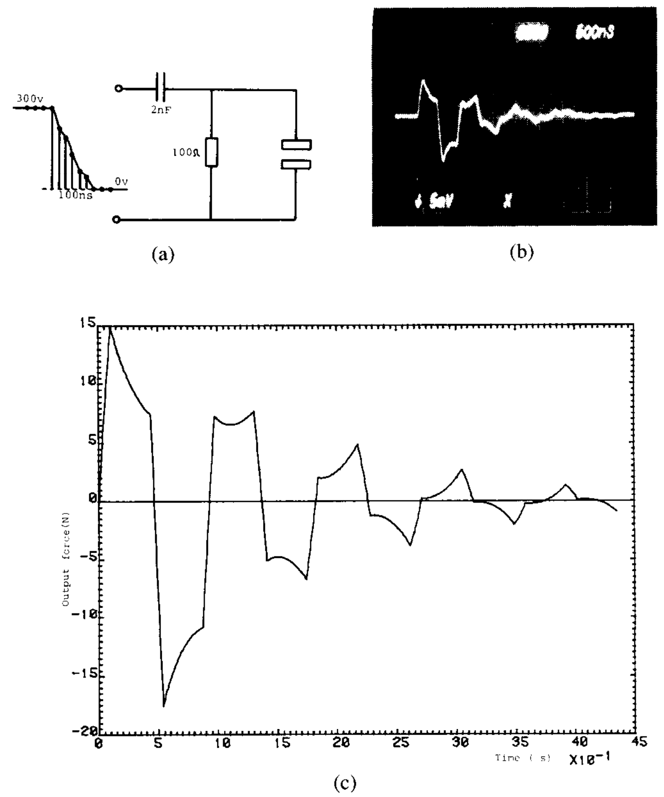

Fig. 11. Figure from [7] showing (a) transducer circuit, (b) experimentally measured force, and (c) simulated force.

V1 1 2
E1 2 0 LAPLACE $\{I(V2)\} = \{"h"/s\}$
V2 E 3
C0 3 0 $"C_o"$
F1 0 3 V1 $"hC_o"$
.ENDS

Fig. 8 also shows a SPICE implementation of the side-electrode bar transducer. The transmission line is defined by its characteristic impedance $Z_o = \rho A_z u_p$ and its time delay $\tau_d = \ell_z/u_p$, where $u_p = [(c - \epsilon h^2)/\rho]^{1/2}$ is the phase velocity. The SPICE code is given below.

*SIDE-ELECTRODE BAR TRANSDUCER SUBCIRCUIT
.SUBCKT XTRN2 E B F
T1 B 1 F 1 Z0 = $"Z_o"$ TD = $"\tau_d"$
V1 1 2
E1 2 0 E 0 $"hC_oA_z/A_x"$
V2 E 3
C0 3 0 $"C_o"$
F1 0 3 V1 $"hC_o"$
.ENDS

V. Example SPICE Simulation

To illustrate an application of the models that have been described, a SPICE simulation of one of the transducers described in [7] is given here. The numerical values for the simulation are the ones given in [9] where a SPICE simulation of the same transducer based on a direct implementation of Mason's model is presented. Fig. 9 shows the circuit diagram for the simulation with the SPICE nodes labeled. The input signal is a voltage pulse with an initial value of 300 V, a final value of 0 V, and a fall time of 100 ns. The transducer is a thickness-mode transducer with the following parameters: diameter $d = 20$ mm, area $A_z = \pi(d/2)^2 = 3.14 \times 10^{-4}\ m^2$, piezoelectric constant $h = 2.15 \times 10^9\ N \cdot m^4/C$, electrode capacitance $C_o = 1.23\ nF$, transmission line characteristic impedance $Z_o = 10.58 \times 10^3$ mech Ω, transmission line time delay $\tau_d = 430\ ns$. The mechanical load on the back of the transducer is assumed to be epoxy resin with a mechanical resistance $R_B = (9.1 \times 10^6$ rayls$) \times A_z = 2860$ mech Ω. The mechanical load on the front is assumed to be water with a mechanical resistance $R_F = (1.5 \times 10^6$ rayls$) \times A_z = 471$ mech Ω. The SPICE code for the simulation is given below.

```
TRANSDUCER SIMULATION
VIN   1 0  PULSE (300,0,0,100N)
C1    1 2  2N
R1    2 0  100
RB    3 0  2.86K
RF    4 0  471
X1    2 3  4  XTRN1A
.TRAN  10N  5U  0N  10N
*THICKNESS-MODE TRANSDUCER SUBCIRCUIT A
.SUBCKT XTRN1A E B F
T1 B 1 F 1 Z0 = 10.58K TD = 430N
V1 1 2
E1 2 0 4 0 1
V2 E 3
C0 3 0 1.23N
F1 0 3 V1 2.65
F2 0 4 V2 2.15E9
R1 4 0 1E3
C1 4 0 1
.ENDS
.PROBE
.END
```

Fig. 10 shows the calculated transient response of the transducer. The waveform (which is inverted) gives the calculated output force as a function of time. Fig. 11 shows the experimentally measured and simulated responses of the transducer from [7]. It can be seen that the waveform of Fig. 10 exhibits excellent agreement with those in Fig. 11. (There is a minor difference between the vertical scales in Figs. 10 and Fig. 11(c). This is because some of the device parameters from [9] were derived from published data because they were otherwise unavailable.)

VI. Conclusion

Derivations of controlled-source analogous circuits for two elementary piezoelectric transducers have been given. The derivations make use of the analogs between the electromechanical equations for the transducer elements and the telegraphist's equations for the voltage and current waves on an electrical transmission line. Unlike traditional models, the models which have been obtained contain no transformers or negative impedance elements. It is believed that these transducer models are more intuitive and straightforward to use than traditional ones. SPICE implementations of each model have been described. These make it possible to calculate the response of a transducer for a wide variety of input functions that include both sinusoidal and arbitrary pulse signals.

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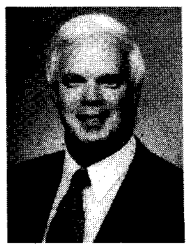

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