Stability Analysis: Bode Plots versus Root Locus

Amit Patel, *Student Member, IEEE,* and Gabriel A. Rincón-Mora, *Senior Member, IEEE*
Georgia Tech Analog, Power, and Energy IC Research

Abstract: In negative feedback, unstable conditions arise when closed-loop gain A_{CL} approaches ∞, that is, when loop gain LG_{FB} is –1 (i.e., $|LG_{FB}| = 1$ and $ang(LG_{FB}) = 180^O$), or more to the point, when $1 + LG_{FB}$ is 0. To ascertain when this occurs and the propensity for a system to become oscillate and/or latch, Bode plots illustrate $|LG_{FB}|$ and $ang(LG_{FB})$ across frequency and Root Locus maps the values of s (which relate to unity-gain freq. f_{0dB}) that produce damped and undamped oscillations as $|LG_{FB}|$ increases from 0 to ∞.

I. Bode Plot

A linear system's closed-loop gain A_{CL} is

$$A_{CL} = \frac{A_{OL}}{1+LG_{FB}} = \frac{A_{OL}}{1+LG_{FB0}\left(\frac{Zeros_{OL}}{Poles_{OL}}\right)}, \tag{1}$$

where LG_{FB0} is loop gain LG_{FB} (i.e., gain through the loop) at low frequencies and $Zeros_{OL}$ and $Poles_{OL}$ refer to the product of zeros and poles present in LG_{FB}. A_{CL} approaches ∞ when its denominator is 0, that is, when

$$1+LG_{FB}=0 \quad \text{or} \quad LG_{FB}=-1, \tag{2}$$

which should be avoided for stable conditions to remain true. In other words, LG_{FB}'s phase shift at unity-gain frequency f_{0dB} should not equal (or exceed) 180^O, which means the propensity for instability depends on how much margin exists before $ang(LG_{FB})$ reaches 180^O (i.e., phase margin PM).

Example:
Consider the following 3-pole, 2-zero system:

$$LG_{FB} = \frac{LG_{FB0}\left(1+\frac{s}{10k}\right)\left(1+\frac{s}{10k}\right)}{\left(1+\frac{s}{10}\right)\left(1+\frac{s}{1k}\right)\left(1+\frac{s}{1k}\right)}. \tag{3}$$

The Bode plot (shown in Fig. 1) illustrates $|LG_{FB}|$ and $ang(LG_{FB})$ across frequency.

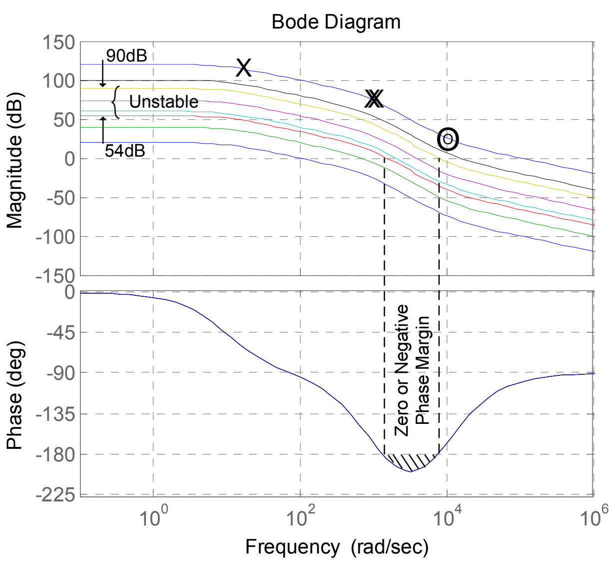

Fig. 1: Bode Plot of a 3-pole, 2-zero system for various values of low-frequency gain LG_{FB0}.

Result: The circuit is unstable if $ang(LG_{FB})$ shifts more than 180^O at LG_{FB}'s unity-gain frequency f_{0dB} (i.e., PM ≤ 0), which in the example examined occurs when LG_{FB0} is 54–90 dB.

II. Root Locus

Root "Locus" refers to the "location" of roots in closed-loop gain A_{CL} (otherwise known as "closed-loop poles") for all possible values of low-frequency loop gain LG_{FB0}. In other words, Root Locus maps all s values that allow LG_{FB} to be –1:

$$1+LG_{FB0}\left(\frac{Zeros_{OL}}{Poles_{OL}}\right)=0, \tag{4}$$

which reduces to

$$Poles_{OL}+LG_{FB0}Zeros_{OL}=0. \tag{5}$$

As a result, Root Locus illustrates how closed-loop poles shift as LG_{FB0} increases from 0 to ∞ (across the σ-jω plane, where s = σ + jω): from open-loop poles $Poles_{OL}$ (when LG_{FB} is low) to open-loop zeros $Zeros_{OL}$ (when $LG_{FB0}Zeros_{OL} >> Poles_{OL}$). Imaginary axis jω indicates oscillating frequencies (i.e., f_{0dB} w/ PM ≤ 0) and real axis σ indicates the amplitude's growth rates (i.e., PM), where negative values indicate decay rates.

Example:
Considering the same example, Root Locus (shown in Fig. 2) illustrates the closed-loop poles' respective trajectories across the σ-jω plane as LG_{FB0} increases.

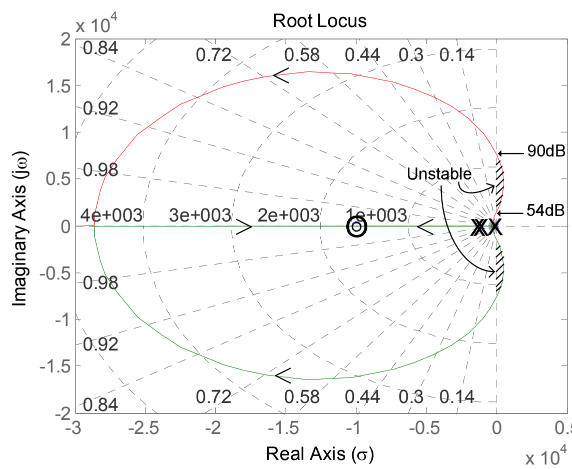

Fig. 2: Root Locus of a 3-pole, 2-zero system.

Result: The circuit is unstable when oscillating frequencies (f_{0dB}'s with PM ≤ 0) self sustain through time (like an oscillator) or grow (i.e., σ ≥ 0), in other words, when a trajectory crosses into the right half plane (RHP) of the σ-jω plot, which in the example examined occurs when LG_{FB0} is 54–90 dB.

Note: Trajectory points with large negative σ values (i.e., fast decay rates) indicate oscillations damp quickly, which translates to more stable and quick-responding (i.e., high-bandwidth) conditions.