EE6604
Personal & Mobile Communications

Week 11
Continuous Phase Modulation
Continuous Phase Modulation (CPM)

- The CPM bandpass signal is

\[ s(t) = \text{Re}\{Ae^{j\phi(t)}e^{j2\pi f_c t}\} = A\cos(2\pi f_c t + \phi(t)) \]  

(1)

where the excess phase is

\[ \phi(t) = 2\pi h \int_0^t \sum_{k=0}^\infty x_k h_f(\tau - kT) d\tau \]

- h is the modulation index
- \( x_n \in \{\pm 1, \pm 3, \ldots, \pm (M-1)\} \) are the M-ary data symbols
- \( h_f(t) \) is the “frequency shaping pulse” of duration \( LT \), that is zero for \( t < 0 \) and \( t > LT \), and normalized to have an area equal to 1/2. Full response CPM has \( L = 1 \), while partial response CPM has \( L > 1 \).

- The instantaneous frequency deviation from the carrier is

\[ f_{\text{dev}}(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} = h \sum_{k=0}^\infty x_k h_f(t - kT) \]
## Frequency Shaping Pulses

<table>
<thead>
<tr>
<th>pulse type</th>
<th>$h_f(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$-rectangular (LREC)</td>
<td>$\frac{1}{2LT} u_{LT}(t)$</td>
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<tr>
<td>$L$-raised cosine (LRC)</td>
<td>$\frac{1}{2LT} \left[ 1 - \cos \left( \frac{2\pi t}{LT} \right) \right] u_{LT}(t)$</td>
</tr>
<tr>
<td>$L$-half sinusoid (LHS)</td>
<td>$\frac{\pi}{4LT} \sin(\pi t / LT) u_{LT}(t)$</td>
</tr>
<tr>
<td>$L$-triangular (LTR)</td>
<td>$\frac{1}{LT} \left( 1 - \frac{</td>
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Excess Phase and Tilted Phase

- During the time interval $nT \leq t \leq (n + 1)T$, the excess phase $\phi(t)$ is

  $$\phi(t) = 2\pi h \sum_{k=0}^{n} x_k \beta(t - kT).$$

  where the “phase shaping pulse” is

  $$\beta(t) = \begin{cases} 
  0, & t < 0 \\
  \int_{0}^{t} h_f(\tau) d\tau, & 0 \leq t \leq LT \\
  1/2, & t \geq LT
  \end{cases}$$

- For the case of full response CPM ($L = 1$), during the time interval $nT \leq t \leq (n + 1)T$ the “excess phase” is

  $$\phi(t) = \pi h \sum_{k=0}^{n-1} x_k + 2\pi hx_n \beta(t - nT)$$

- During the time interval $nT \leq t \leq (n + 1)T$, the CPM “tilted phase” is

  $$\psi(t) = \pi h \sum_{k=0}^{n-1} x_k + 2\pi hx_n \beta(t - nT) + \pi h(M - 1)t/T$$

  $$= \phi(t) + \pi h(M - 1)t/T$$

- Note that $s(t)$ can be generated by replacing $\phi(t)$ with $\psi(t)$ and $f_c$ by $f_c - h(M - 1)t/2T$ in (1).
Continuous Phase Frequency Shift Keying (CPFSK)

- Continuous phase frequency shift keying (CPFSK) is a special type of CPM that uses the shaping function

\[ h_f(t) = \frac{1}{2T} u_T(t) = \frac{1}{2T} (u(t) - u(t - T)) \]

As a result

\[ \beta(t) = \begin{cases} 
0 & , t < 0 \\
\frac{t}{2T} & , 0 \leq t \leq T \\
1/2 & , t \geq T 
\end{cases} \]

- Since the shaping function is rectangular, the CPFSK excess phase trajectories are linear.
Phase tree of binary CPFSK.
Phase-state diagram of CPM with $h = 1/4$. 
Minimum Shift Keying (MSK)

• MSK is a special case of continuous phase frequency shift keying (CPFSK), where the modulation index $h = \frac{1}{2}$ is used.

• The phase shaping pulse is

$$\beta(t) = \begin{cases} 
0 & , t < 0 \\
t/2T & , 0 \leq t \leq T \\
1/2 & , t \geq T 
\end{cases}$$

• The MSK bandpass waveform is

$$s(t) = A \cos \left( 2\pi f_c t + \frac{\pi}{2} \sum_{k=0}^{n-1} x_k + \frac{t - nT}{2T} \pi x_n \right), \quad nT \leq t \leq (n+1)T$$

• The excess phase on the interval $nT \leq t \leq (n+1)T$ is

$$\phi(t) = \frac{\pi}{2} \sum_{k=0}^{n-1} x_k + \frac{t - nT}{2T} \pi x_n$$

• The tilted phase on the interval $nT \leq t \leq (n+1)T$ is

$$\psi(t) = \phi(t) + \frac{\pi t}{2T}$$

• Combining the above two equations, we have

$$\psi((n+1)T) = \psi(nT) + \frac{\pi}{2} (1 + x_n)$$
excess phase trellis diagram for MSK.
Excess Phase and Tilted Phase, example MSK ($h = 1/2$).
Phase state diagram for MSK signals.
Linearized Representation of MSK

- An interesting representation for MSK waveforms can be obtained by using Laurent’s decomposition to express the MSK complex envelope in the quadrature form

\[ \tilde{s}(t) = A \sum_n b(t - 2nT, x_n) , \]

where

\[ b(t, x_n) = \hat{x}_{2n+1} h_a(t - T) + j\hat{x}_{2n} h_a(t) \]

and where \( x_n = (\hat{x}_{2n+1}, \hat{x}_{2n}) \),

\[ \hat{x}_{2n} = \hat{x}_{2n-1} x_{2n} \tag{2} \]
\[ \hat{x}_{2n+1} = -\hat{x}_{2n} x_{2n+1} \tag{3} \]
\[ \hat{x}_{-1} = 1 \tag{4} \]

and

\[ h_a(t) = \sin \left( \frac{\pi t}{2T} \right) u_{2T}(t) . \]

- The sequences, \( \{\hat{x}_{2n}\} \) and \( \{\hat{x}_{2n+1}\} \), are independent binary symbol sequences taking on elements from the set \( \{-1, +1\} \).

- The symbols \( \hat{x}_{2n} \) and \( \hat{x}_{2n+1} \) are transmitted on the quadrature branches with a half-sinusoid (HS) amplitude shaping pulse of duration \( 2T \) seconds and an offset of \( T \) seconds.
Gaussian MSK (GMSK)

Gaussian Pre-modulation filtered MSK (GMSK).

• With MSK the modulating signal is

\[ x(t) = \frac{1}{2T} \sum_{n=-\infty}^{\infty} x_n u_T(t - nT) \]

• We filter \( x(t) \) with a low-pass filter to remove high frequency content prior to modulation, i.e., \( g(t) = x(t) * h(t) \).

• For GMSK, the low-pass filter transfer function is

\[ H(f) = \exp \left\{ - \left( \frac{f}{B} \right)^2 \ln 2 \right\} \]

where \( B \) is the 3 dB bandwidth.
A rectangular pulse $\text{rect}(t/T) = u_T(t + T/2)$ transmitted through this filter yields the GMSK frequency shaping pulse

$$h_f(t) = \frac{1}{2T} \left[ \frac{2\pi}{\ln 2} (BT) \int_{t/T-1/2}^{t/T+1/2} \exp \left\{ -\frac{2\pi^2 (BT)^2 x^2}{\ln 2} \right\} dx \right]$$

$$= \frac{1}{2T} \left[ Q \left( \frac{t/T - 1/2}{\sigma} \right) - Q \left( \frac{t/T + 1/2}{\sigma} \right) \right] \text{ Correction!}$$

where

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$\sigma^2 = \frac{\ln 2}{4\pi^2 (BT)^2}.$$ 

the total pulse area is $\int_{-\infty}^{\infty} h_f(t) dt = 1/2$ and, therefore, the total contribution to the excess phase for each data symbol is $\pm \pi/2$. 

GMSK frequency shaping pulse for various normalized filter bandwidths $BT$. 
The GMSK phase shaping pulse is

$$\beta(t) = \int_{-\infty}^{t} h_f(t) dt = \frac{1}{2} \left( G \left( \frac{t}{T} + \frac{1}{2} \right) - G \left( \frac{t}{T} - \frac{1}{2} \right) \right)$$

where

$$G(x) = x \Phi \left( \frac{x}{\sigma} \right) + \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}},$$

and

$$\Phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Observe that $\beta(\infty) = 1/2$ and, therefore, the total contribution to the excess phase for each data symbol remains at $\pm \pi/2$. 
**GMSK phase shaping pulse for various normalized filter bandwidths $BT$.**
• The excess phase change over the interval from $-T/2$ to $T/2$ is

$$
\phi(T/2) - \phi(-T/2) = x_0\beta_0(T) + \sum_{n=-\infty}^{\infty} x_n\beta_n(T)
$$

where

$$
\beta_n(T) = \int_{-T/2-nT}^{T/2-nT} h_f(\nu) \, d\nu .
$$

and

$$
h_f(t) = \frac{1}{2T} \left[ Q \left( \frac{t/T - 1/2}{\sigma} \right) - Q \left( \frac{t/T + 1/2}{\sigma} \right) \right]
$$

• The first term, $x_0\beta_0(T)$ is the desired term, and the second term, $\sum_{n=-\infty}^{\infty} x_n\beta_n(T)$, is the intersymbol interference (ISI) introduced by the Gaussian low-pass filter.

• Conclusion: GMSK trades off power efficiency for a greatly improved bandwidth efficiency.
Power spectral density of GMSK with various normalized filter bandwidths $BT$. 

$$S_{vv}(f) (\text{dB})$$

Frequency, $fT$
Linearized Gaussian Minimum Shift Keying (LGMSK)

- Laurent showed that any binary partial response CPM signal can be represented exactly as a linear combination of $2^{L-1}$ partial-response pulse amplitude modulated (PAM) signals, viz.,

$$\tilde{s}(t) = \sum_{n=0}^{\infty} \sum_{p=0}^{2^{L-1}-1} e^{j\pi h_{n,p}} c_p(t - nT),$$

where

$$c_p(t) = c(t) \prod_{n=1}^{L-1} c(t + (n + L\varepsilon_{n,p})T),$$

$$\alpha_{n,p} = \sum_{m=0}^{n} x_m - \sum_{m=1}^{L-1} x_{n-m}\varepsilon_{m,p},$$

and $\varepsilon_{n,p} \in \{0, 1\}$ are the coefficients of the binary representation of the index $p$, i.e.,

$$p = \varepsilon_{0,p} + 2\varepsilon_{1,p} + \cdots + 2^{L-2}\varepsilon_{L-2,p}.$$

The basic signal pulse $c(t)$ is

$$c(t) = \begin{cases} \frac{\sin(2\pi h\beta(t))}{\sin(\pi h)} & , \quad 0 \leq t < LT \\ \frac{\sin(\pi h - 2\pi h\beta(t-LT))}{\sin(2\pi h)} & , \quad LT \leq t < 2LT \\ 0 & , \quad \text{otherwise} \end{cases},$$

where $\beta(t)$ is the CPM phase shaping function.
Linearized Gaussian Minimum Shift Keying (LGMSK)

- note that the GMSK frequency shaping pulse spans approximately $L = 4$ symbol periods for practical values of $BT$.

- Often the pulse $c_0(t)$ contains most of the signal energy, so the $p = 0$ term in can provide a good approximation to the CPM signal. Numerical analysis can show that the pulse $c_0(t)$ contains 99.83% of the energy and, therefore, we can derive a linearized GMSK waveform by using only $c_0(t)$ and neglecting the other pulses.

- This yields the waveform

$$\tilde{s}(t) = \sum_{n=0}^{\infty} e^{j\pi h\alpha_{n,0}} c_0(t - nT),$$

where, with $L = 4$,

$$c_0(t) = \prod_{n=0}^{3} c(t + nT),$$

$$\alpha_{n,0} = \sum_{m=0}^{n} x_m$$

- Since the GMSK phase shaping pulse is non-causal, when evaluating $c(t)$ we use the truncated and time shifted GMSK phase shaping pulse

$$\hat{\beta}(t) = \beta(t - 2T)$$

with $L = 4$ as shown previously.
LGMSK amplitude shaping pulse for various normalized premodulation filter bandwidths $BT$. 
Linearized Gaussian Minimum Shift Keying (LGMSK)

- For $h = 1/2$ used in GMSK,

\[ a_{n,0} = e^{j\frac{\pi}{2}\alpha_{n,0}} \in \{\pm 1, \pm j\} , \]

and it follows that

\[ \tilde{s}(t) = A \sum_n \left( \hat{x}_{2n+1} c_0(t - 2nT - T) + j\hat{x}_{2n} c_0(t - 2nT) \right) \]

where

\[ \hat{x}_{2n} = \hat{x}_{2n-1} x_{2n} \]
\[ \hat{x}_{2n+1} = -\hat{x}_{2n} x_{2n+1} \]
\[ \hat{x}_{-1} = 1 \]

- This is the same as the OQPSK representation for MSK except that the half-sinusoid amplitude pulse shaping function is replaced with the LGMSK amplitude pulse shaping function.

- Note that the LGMSK pulse has length of approximately $4T$, while the pulses on the quadrature branches are transmitted every $2T$ seconds. Therefore, the LGMSK pulse introduces ISI.