1.6. Gaussian Random Variables & Processes

- If $W \sim \mathcal{N}(\mu, \sigma^2)$,

  Then $f_W(w) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(w-\mu)^2}{2\sigma^2}}$

  μ : mean
  σ^2 : Variance

- $$\mu = E[W] = \int_{-\infty}^{\infty} w \, f_W(w)\, dw = \int_{-\infty}^{\infty} w \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(w-\mu)^2}{2\sigma^2}} \cdot dw$$

  $$\sigma^2 = E[(W-\mu)^2] = E[W^2] - \mu^2 = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} w^2 \cdot e^{-\frac{(w-\mu)^2}{2\sigma^2}} dw$$

- Measured signals are corrupted by noise (additive mod

  $$\vec{Z}[t] = \vec{Y}[t] + \underbrace{\vec{W}[t]}_{\text{Vector of noise samples}}$$

  Measured output

- We assume additive noise signals are Gaussian
  - → Quantization is diffrent
  - → Thermal noise is Gaussian

- The distribution of several random variables approximates a Gaussian distribution, i.e,

  $X_1, X_2, \ldots X_N$ are IIDs with means $\mu_1, \mu_2 \ldots \mu_N$ and variances $\sigma_1^2, \sigma_2^2 \ldots, \sigma_N^2$. Then

  $$Y = \sum_{i=1}^{N} \frac{X_i - \mu_i}{\sigma_i} \quad \text{is distributed like } \mathcal{N}(0,1)$$

  when $N \to \infty$, then $Y \sim N(0,1)$

  Proof is by the use of central limit theorem.

- A Gaussian RV W is entirely determined by its mean (μ) and variance (σ^2).

- A Gaussian RP $w(t)$ is determined by its

  mean
  $$m_w(t) = E[w(t)] \quad \&$$

  auto correlation
  $$r_w(t,s) = E[w(t)\,\bar{w}(s)]$$

- A Gaussian RP with constant mean
  $m_w(t) = \text{Constant}$ &
  $r_w(t,s) = r_w(s-t)$ is stationary.

- If X and Y are jointly Gaussian, then
  $Z = aX + bY$
  is also Gaussian. (The sum of Gaussians is a Gaussian)

- If a Gaussian RP ~~w~~ $w(t)$ is input to LTI system, Output is also Gaussian. All that change is mean & ~~v~~ autocorrelation.

- Maximum Likelihood detection/estimation involving Gaussian RVs corresponds to a Euclidean distance.

- WSS (Wide sense stationary) G RP are also SSS (Strict sense stationary)

- Uncorrelated G RV's are also independent

- Gaussian conditioned on a Gaussian is also Gaussian

- If $\vec{W}_{(k\times 1)}$ is a GRV with mean $\vec{\mu}$ & covariance matrix $\vec{R}$, then

$$f_{\vec{W}}(\vec{w}) = \frac{1}{(2\pi)^{k/2}\,|\vec{R}|^{1/2}}\, e^{-\frac{1}{2}(\vec{w}-\vec{\mu})^T \vec{R}^{-1}(\vec{w}-\vec{\mu})}$$

where

$$\vec{\mu} = E[\vec{W}] = \begin{bmatrix} E[w_1] \\ E[w_2] \\ \vdots \\ E[w_k] \end{bmatrix}$$

and

$$\underset{k\times k}{\vec{R}} = E\left[(\vec{W}-\vec{\mu})(\vec{W}-\vec{\mu})^T\right] = E[\vec{W}\vec{W}^T] - \vec{\mu}\vec{\mu}^T$$

- Let's look at the 2-dimensional case

$$\vec{R} = E\left[(\vec{W}-\vec{\mu})(\vec{W}-\vec{\mu})^T\right]$$

$$= E\left[\begin{bmatrix} w_1-\mu_1 \\ w_2-\mu_2 \end{bmatrix}\begin{bmatrix} w_1-\mu_1 & w_2-\mu_2 \end{bmatrix}\right]$$

$$= E\begin{bmatrix} (w_1-\mu_1)^2 & (w_1-\mu_2)(w_2-\mu_2) \\ (w_1-\mu_1)(w_2-\mu_2) & (w_2-\mu_2)^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

- Correlation coefficient is defined as

$$\rho = \frac{\sigma_{12}}{\sigma_1 . \sigma_2} \quad \& \quad -1 \le \rho \le 1.$$

(Rho)

- If $\rho = 1$, then $w_1 = w_2$
  $\rho = -1$, then $w_1 = -w_2$
  $\rho = 0$, then uncorrelated

- $\sigma_{12} = \sigma_1 . \sigma_2 . \rho$

$$\vec{R} = \begin{bmatrix} \sigma_1^2 & \sigma_1 \sigma_2 \rho \\ \sigma_1 \sigma_2 \rho & \sigma_2^2 \end{bmatrix}$$

$$\vec{R}^{-1} = \frac{1}{\sigma_1^2 \sigma_2^2 - \sigma_1^2 \sigma_2^2 \rho^2} \cdot \begin{bmatrix} \sigma_2^2 & -\sigma_1 \sigma_2 \rho \\ -\sigma_1 \sigma_2 \rho & \sigma_1^2 \end{bmatrix}$$

$$= \frac{1}{(1-\rho^2)} \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1 \sigma_2} \\ \frac{-\rho}{\sigma_1 \sigma_2} & \frac{1}{\sigma_2^2} \end{bmatrix}$$

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$$f_{\vec{W}}(\vec{w}) = \frac{1}{(2\pi)^{2/2} \cdot |R|^{1/2}} \, e^{-\frac{1}{2}\left[w_1-\mu_1 \quad w_2-\mu_2\right] \vec{R}^{-1} \left[w_1-\mu_1 \quad w_2-\mu_2\right]^T}$$

$$= \frac{1}{2\pi \cdot \sigma_1\sigma_2\sqrt{(1-\rho^2)}} \, e^{-\frac{1}{2}\left[w_1-\mu_1 \quad w_2-\mu_2\right] \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ -\frac{\rho}{\sigma_1\sigma_2} & 1/\sigma_2^2 \end{bmatrix} \cdot \frac{1}{1-\rho^2}}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)} \cdot \left\{ \frac{(w_1-\mu_1)^2}{\sigma_1^2} + \frac{(w_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(w_1-\mu_1)(w_2-\mu_2)}{\sigma_1\sigma_2} \right\}}$$

- If $\rho = 0$,

$$f_{\vec{W}}(w_1, w_2) = \frac{1}{2\pi\sigma_1\sigma_2} \, e^{-\frac{1}{2}\left[\frac{(w_1-\mu_1)^2}{\sigma_1^2} + \frac{(w_2-\mu_2)^2}{\sigma_2^2}\right]}$$

$$= f_{W_1}(w_1) \cdot f_{W_2}(w_2) \quad \text{[Independent]}.$$

Conditional Gaussian Densities

- Suppose X & Y are jointly Gaussian with ρ

  $X \sim N(\mu_x, \sigma_x^2)$

  $Y \sim N(\mu_y, \sigma_y^2)$

- We want to find an estimate $\hat{x}$ for X, given $Y=y$

Without any knowledge,

$$\hat{x} = \mu_x$$

But we know $Y=y$ & X&Y are jointly G.

$$\hat{x} = E[f_{X|Y}(x|y)]$$

Then, we ~~fin~~ need to find $f_{X|Y}(x|y)$

$$f_{X|Y}(x|Y=y) = \frac{f(x,y)}{f_Y(y)}$$

$$f_{X|Y}(x|y) = \frac{\dfrac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\cdot\sigma_y}\right\}}}{\dfrac{1}{\sqrt{2\pi}\,\sigma_y}\cdot e^{-\frac{1}{2}\cdot\frac{(y-\mu_y)^2}{\sigma_y^2}}}$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}\cdot\sigma_x}\, e^{-\frac{1}{2\sigma_x^2(1-\rho^2)}\left[x-\left(\mu_x+\frac{\sigma_x}{\sigma_y}\rho(y-\mu_y)\right)\right]^2}$$

- $f_{X|Y}$ is also Gaussian with

$$E[X|Y=y] = \mu_x + \frac{\sigma_x}{\sigma_y}\rho(y-\mu_y)$$

$$Var(X|Y=y) = \sigma_x^2(1-\rho^2)$$

- $\hat{x} = \mu_x + \frac{\sigma_x}{\sigma_y}\rho(y-\mu_y).$

Homework

- Read pp. 71–121
- Solve the problems (Due date: Sept 8, 2004)

  1.4.3
  1.4.7
  1.4.8.
  1.4.10
  1.4.13
  1.4.15
  1.6.34.
  1.6.35
  1.6.37
- Only one problem will be graded.
- Send your track selection (by e-mail) by Sept 8.
  cc to TA