Derivation of the Drift-Diffusion Equation

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Drift-Diffusion Equation - Applicability

Basic equations governing transport in semiconductors and semiconductor devices:

- Instances where Drift-Diffusion Equation cannot be used
  - Accelerations during rapidly changing electric fields (transient effects)
    - Non quasi-steady state
    - Non-Maxwellian distribution
  - Accurate prediction of the distribution or spread of the transport behavior is required
- Instances when Drift-Diffusion Equation can represent the trend (or predict the mean behavior of the transport properties)
  - Feature length of the semiconductors smaller than the mean free path of the carriers
- Instances when Drift-Diffusion equations are accurate
  - Quasi-steady state assumption holds (no transient effects)
  - Device feature lengths much greater than the mean free paths of the carrier

The Method of Moments

In the Method of Moments, both sides of a equation are multiplied by a function (a moment generating function) raised to a integer power, and then integrated over all space

$$\therefore \quad A(x,k,t) + B(x,k,t) = C(x,k,t)$$

↓ Multiplying by the Moment generating Function $\Theta^n(k)$ (n = order of the moment)

$$\int \Theta^k(k) A(x,k,t) d^3k + \int \Theta^k(k) B(x,k,t) d^3k = \int \Theta^k(k) C(x,k,t) d^3k$$

Method of Moments Applied to the Boltzmann Transport Equation

The Boltzmann Transport Equation with relaxation time approximation:

$$\frac{\partial f}{\partial t} + \frac{\vec{F}_{ext}}{\hbar} \cdot \vec{\nabla}_k f + \vec{v} \cdot \vec{\nabla}_x f = -\frac{f - f_0}{\tau}$$

f = a classical distribution function at nonequilibrium state that represents the probability of finding a particle at position x, with momentum k and at time t. The subscript 0 corresponds to the equilibrium state

Multiplying throughout by the moment generating function Θ^n and integrating over all k space

$$\int \Theta^n \frac{\partial f}{\partial t} d^3k + \frac{1}{\hbar} \int \Theta^n \left(\vec{F}_{ext} \cdot \vec{\nabla}_k f \right) d^3k + \int \Theta^n \left(\vec{v} \cdot \vec{\nabla}_x f \right) d^3k = -\int \Theta^n \frac{f - f_0}{\tau} d^3k$$

Method of Moments Applied to the Boltzmann Transport Equation

If $\Theta = 1$ and n = 1 then:

$$\int \frac{\partial f}{\partial t} d^3k + \frac{1}{\hbar}\int \left(\vec{F}_{ext} \cdot \vec{\nabla}_k f\right) d^3k + \int \left(\vec{v} \cdot \vec{\nabla}_x f\right) d^3k = -\int \frac{f - f_0}{\tau} d^3k$$

Simplifies to

$$\frac{\partial n}{\partial t} + \vec{\nabla}_x \cdot \left(n\vec{v}\right) = 0$$

→ Carrier Continuity Equation

If $\Theta = v$ and n = 1 then:

$$\int \vec{v}\frac{\partial f}{\partial t} d^3k + \frac{1}{\hbar}\int \vec{v}\left(\vec{F}_{ext} \cdot \vec{\nabla}_k f\right) d^3k + \int \vec{v}\left(\vec{v} \cdot \vec{\nabla}_x f\right) d^3k = -\int \vec{v}\frac{f - f_0}{\tau} d^3k$$

Simplifies to

$$J_n = nq\mu_n \vec{F} + qD_n \vec{\nabla}_x n$$

→ Drift-Diffusion Equation

In the subsequent slides we would derive the Drift-Diffusion Equation from Boltzmann Transport Equation by utilizing this Method of Moments

Drift-Diffusion Equation Derivation – 1st. Term

$$\int \vec{v}\frac{\partial f}{\partial t}d^3k + \frac{1}{\hbar}\int \vec{v}\left(\vec{F}_{ext}\cdot\vec{\nabla}_k f\right)d^3k + \int \vec{v}\left(\vec{v}\cdot\vec{\nabla}_x f\right)d^3k = -\int \vec{v}\frac{f-f_0}{\tau}d^3k$$

Velocity is time independent

$$\frac{\partial}{\partial t}\int \vec{v} f d^3k$$

$$n = carrier \quad concentration = \int G(k) f(k) d^3k$$

$$G(k) = Density \quad of \quad states$$

$$= \frac{1}{V}\frac{dN}{dk} = \frac{2}{(2\pi)^3} = \frac{1}{4\pi^3}$$

$$\therefore \quad n = \int \frac{1}{4\pi^3} f(k) d^3k = \int f'(k) d^3k$$

$$\int \vec{v}\frac{\partial f}{\partial t}d^3k = \frac{\partial}{\partial t}(n\vec{v})$$

Drift-Diffusion Equation Derivation – 2nd. Term

$$\int \vec{v}\frac{\partial f}{\partial t}d^3k + \frac{1}{\hbar}\int \vec{v}\left(\vec{F}_{ext}\cdot\vec{\nabla}_k f\right)d^3k + \int \vec{v}\left(\vec{v}\cdot\vec{\nabla}_x f\right)d^3k = -\int \vec{v}\frac{f-f_0}{\tau}d^3k$$

$$Identity:\quad \vec{F}\cdot\vec{\nabla}g = \vec{\nabla}\cdot\left(g\vec{F}\right) - g\vec{\nabla}\cdot\vec{F}$$

$$\frac{1}{\hbar}\int \vec{v}\left(\vec{\nabla}_k\cdot f\vec{F}_{ext}\right)d^3k - \frac{1}{\hbar}\int f\vec{v}\left(\vec{\nabla}_k\cdot\vec{F}_{ext}\right)d^3k$$

$$Identity:\quad \left(g\vec{F}\cdot\vec{\nabla}\right)\vec{G} = \vec{\nabla}\cdot\left(g\vec{F}\vec{G}\right) - \vec{G}\vec{\nabla}\cdot\left(g\vec{F}\right)$$

$$\frac{1}{\hbar}\left\{\int\left[\vec{\nabla}_k\left(f\vec{v}\vec{F}_{ext}\right) - \left(f\vec{F}_{ext}\cdot\vec{\nabla}_k\right)\vec{v}\right]d^3k\right\}$$

f is finite and so the surface integral (integral of divergence of fvF_{ext}) at infinity vanishes identically

$$-\frac{1}{\hbar}\int\left(f\vec{F}_{ext}\cdot\vec{\nabla}_k\right)\vec{v}\,d^3k$$

Drift-Diffusion Equation Derivation – 2nd. Term (Continued)

Substituting:

$$-\frac{1}{\hbar}\int\left(f\vec{F}_{ext}\cdot\vec{\nabla}_k\right)\vec{v}\,d^3k-\frac{1}{\hbar}\int f\vec{v}\left(\vec{\nabla}_k\cdot\vec{F}_{ext}\right)d^3k$$

Substituting $-\vec{\nabla}_x E=\vec{F}_{ext}$

$$\frac{1}{\hbar}\int\left(f\vec{\nabla}_x E\cdot\vec{\nabla}_k\right)\vec{v}\,d^3k+\frac{1}{\hbar}\int f\vec{v}\left(\vec{\nabla}_k\cdot\vec{\nabla}_x E\right)d^3k$$

$$\vec{\nabla}_k\cdot\vec{\nabla}_x E=\vec{\nabla}_x\cdot\vec{\nabla}_k E=\hbar\vec{\nabla}_x\cdot\vec{v}$$

$$\int f\vec{v}\left(\vec{\nabla}_x\cdot\vec{v}\right)d^3k$$

Substituting, the second term is finally reduced to:

$$\frac{1}{\hbar}\int\vec{v}\left(\vec{F}_{ext}\cdot\vec{\nabla}_k f\right)d^3k=\frac{1}{\hbar}\int\left(f\vec{\nabla}_x E\cdot\vec{\nabla}_k\right)\vec{v}\,d^3k+\int f\vec{v}\left(\vec{\nabla}_x\cdot\vec{v}\right)d^3k$$

Drift-Diffusion Equation Derivation – 3rd. Term

$$\int \vec{v}\frac{\partial f}{\partial t}d^3k + \frac{1}{\hbar}\int \vec{v}\left(\vec{F}_{ext}\cdot\vec{\nabla}_k f\right)d^3k + \int \vec{v}\left(\vec{v}\cdot\vec{\nabla}_x f\right)d^3k = -\int \vec{v}\,\frac{f-f_0}{\tau}d^3k$$

$$Identity:\quad \vec{F}\cdot\vec{\nabla}g = \vec{\nabla}\cdot\left(g\vec{F}\right) - g\vec{\nabla}\cdot\vec{F}$$

$$\int \vec{v}\vec{\nabla}_x\cdot\left(f\vec{v}\right)d^3k - \int \vec{v}f\left(\vec{\nabla}_x\cdot\vec{v}\right)d^3k$$

$$Identity:\quad \vec{G}\vec{\nabla}\cdot\left(g\vec{F}\right) = \vec{\nabla}\cdot\left(g\vec{F}\vec{G}\right) - \left(g\vec{F}\cdot\vec{\nabla}\right)\vec{G}$$

$$\int \vec{\nabla}_x\cdot\left(f\vec{v}\vec{v}\right)d^3k - \int \left(f\vec{v}\cdot\vec{\nabla}_x\right)\vec{v}\,d^3k$$

Substituting, the third term is finally reduced to:

$$\int \vec{v}\left(\vec{v}\cdot\vec{\nabla}_x f\right)d^3k = \int \vec{\nabla}_x\cdot\left(f\vec{v}\vec{v}\right)d^3k - \int \left(f\vec{v}\cdot\vec{\nabla}_x\right)\vec{v}\,d^3k - \int \vec{v}f\left(\vec{\nabla}_x\cdot\vec{v}\right)d^3k$$

Drift-Diffusion Equation Derivation – Right Hand Term

$$\int \vec{v}\frac{\partial f}{\partial t}d^3k + \frac{1}{\hbar}\int \vec{v}\left(\vec{F}_{ext}\cdot\vec{\nabla}_k f\right)d^3k + \int \vec{v}\left(\vec{v}\cdot\vec{\nabla}_x f\right)d^3k = -\int \vec{v}\frac{f-f_0}{\tau}d^3k$$

$$-\frac{1}{\tau}\int \vec{v}\left(f-f_0\right)d^3k$$

$$\text{Recall} \quad \int f d^3k = n \quad and \int \vec{v} f d^3k = \bar{v}n$$

n = carrier concentration
v = average velocity

$$-n\frac{\bar{v}-\bar{v}_0}{\tau}$$

At equilibrium the ensemble velocity v_0 (by definition) = 0

Substituting, the right hand term is finally reduced to:

$$-\int \vec{v}\frac{f-f_0}{\tau}d^3k = -n\frac{\bar{v}}{\tau}$$

Drift-Diffusion Equation Derivation – General Form

$$\int \vec{v}\,\frac{\partial f}{\partial t}\,d^3k + \frac{1}{\hbar}\int \vec{v}\left(\vec{F}_{ext}\cdot\vec{\nabla}_k f\right)d^3k + \int \vec{v}\left(\vec{v}\cdot\vec{\nabla}_x f\right)d^3k = -\int \vec{v}\,\frac{f-f_0}{\tau}\,d^3k$$

$$\frac{\partial(n\bar{v})}{\partial t} + \frac{1}{\hbar}\int\left(f\vec{\nabla}_x E\cdot\vec{\nabla}_k\right)\vec{v}\,d^3k + \int f\vec{v}\left(\vec{\nabla}_x\cdot\vec{v}\right)d^3k +$$

$$\int \vec{\nabla}_x\cdot\left(f\vec{v}\vec{v}\right)d^3k - \int\left(f\vec{v}\cdot\vec{\nabla}_x\right)\vec{v}\,d^3k - \int \vec{v}f\left(\vec{\nabla}_x\cdot\vec{v}\right)d^3k = -n\frac{\bar{v}}{\tau}$$

$$\frac{\partial(n\bar{v})}{\partial t} + \frac{1}{\hbar}\int\left(f\vec{\nabla}_x E\cdot\vec{\nabla}_k\right)\vec{v}\,d^3k + \int \vec{\nabla}_x\cdot\left(f\vec{v}\vec{v}\right)d^3k - \int\left(f\vec{v}\cdot\vec{\nabla}_x\right)\vec{v}\,d^3k = -n\frac{\bar{v}}{\tau}$$

Standard Drift-Diffusion Equation for Electrons/Holes

The general Drift-Diffusion derived in the previous slides may be further simplified with the help of certain assumptions

- Assumptions
  - The energy of the carriers, $E = \dfrac{\hbar^2 k^2}{2m}$
  - Mass is isotropic and constant

$$\therefore \quad E_x = E_y = E_z = E_i = \frac{1}{2} m v_i^2$$

  - Material is isotropic, and so the spatial temperature gradient is zero $\vec{\nabla}_x E_i = 0$

Standard Drift-Diffusion Equation for Electrons/Holes-Text Version

$$\frac{\partial(n\bar{v})}{\partial t}+\boxed{\frac{1}{\hbar}\int\left(f\vec{\nabla}_x E\cdot\vec{\nabla}_k\right)\vec{v}\,d^3k}+\int\vec{\nabla}_x\cdot\left(f\vec{v}\vec{v}\right)d^3k-\int\left(f\vec{v}\cdot\vec{\nabla}_x\right)\vec{v}\,d^3k=-n\frac{\bar{v}}{\tau}$$

$$Substituting\quad -\vec{\nabla}_x E=\vec{F}_{ext}$$

$$\boxed{-\frac{1}{\hbar}\int f\vec{F}_{ext}\cdot\vec{\nabla}_k\vec{v}\,d^3k}$$

$$E=\frac{\hbar^2k^2}{2m}\Rightarrow\vec{\nabla}_k E=\frac{\hbar^2\vec{k}}{m}\xrightarrow{\vec{k}=m\vec{v}}\vec{v}=\frac{1}{\hbar}\vec{\nabla}_k E\Rightarrow\vec{\nabla}_k\vec{v}=\frac{1}{\hbar}\vec{\nabla}_k^2 E$$

$$again\quad E=\frac{\hbar^2k^2}{2m}\Rightarrow\vec{\nabla}_k^2 E=\frac{\hbar^2}{m}$$

$$\therefore\quad\vec{\nabla}_k\vec{v}=\frac{\hbar}{m}$$

$$\boxed{-\frac{\vec{F}_{ext}}{m}\int f d^3k}\xrightarrow{recall\ \int f d^3k=n}\boxed{-\frac{\vec{F}_{ext}}{m}n}$$

Standard Drift-Diffusion Equation for Electrons/Holes-Text Version

$$\frac{\partial(n\bar{v})}{\partial t}+\frac{1}{\hbar}\int\left(f\vec{\nabla}_x E\cdot\vec{\nabla}_k\right)\vec{v}d^3k+\boxed{\int\vec{\nabla}_x\cdot\left(f\vec{v}\vec{v}\right)d^3k}-\int\left(f\vec{v}\cdot\vec{\nabla}_x\right)\vec{v}d^3k=-n\frac{\bar{v}}{\tau}$$

Assuming the mass is isotropic and constant and therefore:

$$f\vec{v}\vec{v}=f\begin{bmatrix}v_x^2 & 0 & 0\\ 0 & v_y^2 & 0\\ 0 & 0 & v_z^2\end{bmatrix}=\frac{2}{m}\begin{bmatrix}fE_x & 0 & 0\\ 0 & fE_y & 0\\ 0 & 0 & fE_z\end{bmatrix}\quad taking\quad E_x=E_y=E_z=E_i=\frac{1}{2}mv_i^2$$

$$now\quad \vec{\nabla}_x\cdot\left(f\vec{v}\vec{v}\right)=\frac{2}{m}\left[\frac{\partial fE_x}{\partial x}+\frac{\partial fE_y}{\partial y}+\frac{\partial fE_z}{\partial z}\right]=\frac{2}{m}\left[f\vec{\nabla}_x E_i+E_i\vec{\nabla}_x f\right]$$

$$\therefore\quad \int\vec{\nabla}_x\cdot\left(f\vec{v}\vec{v}\right)d^3k=\frac{2}{m}\left[n\vec{\nabla}_x E_i+E_i\vec{\nabla}_x n\right]$$

Assuming the material is isotropic i.e. temperature or energy is spatially independent

$$\therefore\quad \boxed{\int\vec{\nabla}_x\cdot\left(f\vec{v}\vec{v}\right)d^3k=\frac{2}{m}\left[\cancelto{0}{n\vec{\nabla}_x E_i}+E_i\vec{\nabla}_x n\right]\xrightarrow{E_i=\frac{1}{3}\bar{E}}\frac{2}{3m}\bar{E}\vec{\nabla}_x n}$$

Standard Drift-Diffusion Equation for Electrons/Holes-Text Version

$$\frac{\partial(n\bar{v})}{\partial t}+\frac{1}{\hbar}\int\left(f\vec{\nabla}_x E\cdot\vec{\nabla}_k\right)\vec{v}\,d^3k+\int\vec{\nabla}_x\cdot\left(f\vec{v}\vec{v}\right)d^3k-\int\left(f\vec{v}\cdot\vec{\nabla}_x\right)\vec{v}\,d^3k=-n\frac{\bar{v}}{\tau}$$

Notice that this term is completely ignored in the text

$$\frac{\partial(n\bar{v})}{\partial t}-\frac{\vec{F}_{ext}}{m}n+\frac{2}{3m}\bar{E}\vec{\nabla}_x n=-n\frac{\bar{v}}{\tau}$$

$\vec{F}_{ext}=-q\vec{F}$ *where* $q=$ *electronic charge,* $\vec{F}=$ *field*

$\bar{E}=\frac{3}{2}k_BT,$ *where* $T=$ *temperature*

$$\tau\frac{\partial(-qn\bar{v})}{\partial t}+(-qn\bar{v})=\frac{q^2\tau\vec{F}}{m}n+\frac{q\tau}{m}k_BT\vec{\nabla}_x n$$

electron current density $=J_n=-qn\bar{v}$

electron mobility $=\mu_n=\frac{q\tau}{m}$

$$\tau\frac{\partial J_n}{\partial t}+J_n=nq\mu_n\vec{F}+\mu_n k_BT\vec{\nabla}_x n$$

Standard Drift-Diffusion Equation for Electrons/Holes-My Version

$$\frac{\partial(n\bar{v})}{\partial t}+\frac{1}{\hbar}\int\left(f\vec{\nabla}_x E\cdot\vec{\nabla}_k\right)\vec{v}\,d^3k+\boxed{\int\vec{\nabla}_x\cdot\left(f\vec{v}\vec{v}\right)d^3k-\int\left(f\vec{v}\cdot\vec{\nabla}_x\right)\vec{v}\,d^3k}=-n\frac{\bar{v}}{\tau}$$

$$Identity:\quad \vec{G}\vec{\nabla}\cdot\left(g\vec{F}\right)=\vec{\nabla}\cdot\left(g\vec{F}\vec{G}\right)-\left(g\vec{F}\cdot\vec{\nabla}\right)\vec{G}$$

$$\boxed{\int\vec{v}\vec{\nabla}_x\cdot\left(f\vec{v}\right)d^3k}$$

$$Identity:\quad \vec{\nabla}\cdot\left(g\vec{F}\right)=\vec{F}\cdot\vec{\nabla}g+g\vec{\nabla}\cdot\vec{F}$$

$$\boxed{\int\vec{v}\left[f\vec{\nabla}_x\cdot\vec{v}\right]d^3k+\int\vec{v}\left[\vec{v}\cdot\vec{\nabla}_x f\right]d^3k}$$

$$\text{Re}\,call\quad \vec{\nabla}_x\cdot\vec{v}=\frac{1}{\hbar}\vec{\nabla}_x\cdot\vec{\nabla}_k E=\frac{1}{\hbar}\vec{\nabla}_k\cdot\vec{\nabla}_x E$$

$$\boxed{\int\vec{v}\left[f\frac{1}{\hbar}\vec{\nabla}_k\cdot\underbrace{\vec{\nabla}_x E}_{0}\right]d^3k+\int\vec{v}\left[\vec{v}\cdot\vec{\nabla}_x f\right]d^3k}\Rightarrow \text{Next Slide}$$

Assuming the material is isotropic
i.e. energy is spatially independent $\vec{\nabla}_x E=0$

Standard Drift-Diffusion Equation for Electrons/Holes-My Version

Previous Slide →

$$\int \vec{v}\vec{v}\cdot\vec{\nabla}_x f d^3k$$

Assuming the mass is isotropic and constant and therefore:

$$\vec{v}\vec{v} = \begin{bmatrix} v_x^2 & 0 & 0 \\ 0 & v_y^2 & 0 \\ 0 & 0 & v_z^2 \end{bmatrix} = \frac{2}{m}\begin{bmatrix} E_x & 0 & 0 \\ 0 & E_y & 0 \\ 0 & 0 & E_z \end{bmatrix} \quad taking \quad E_x = E_y = E_z = E_i = \frac{1}{2}mv_i^2$$

$$now \quad \vec{v}\vec{v}\cdot\vec{\nabla}_x f = \frac{2E_i}{m}\left[\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\right] = \frac{2}{m}\left[E_i\vec{\nabla}_x f\right]$$

$$\therefore \quad \int \vec{v}\vec{v}\cdot\vec{\nabla}_x f d^3k = \frac{2}{m}\left[E_i\vec{\nabla}_x n\right]$$

$$\therefore \int \vec{v}\vec{v}\cdot\vec{\nabla}_x f d^3k = \frac{2}{m}\left[E_i\vec{\nabla}_x n\right] \xrightarrow{E_i=\frac{1}{3}\overline{E}} \frac{2}{3m}\overline{E}\vec{\nabla}_x n$$

Standard Drift-Diffusion Equation for Electrons/Holes-My Version

$$\frac{\partial(n\bar{v})}{\partial t}+\frac{1}{\hbar}\int\left(f\vec{\nabla}_x E\cdot\vec{\nabla}_k\right)\vec{v}\,d^3k+\int\vec{\nabla}_x\cdot\left(f\vec{v}\vec{v}\right)d^3k-\int\left(f\vec{v}\cdot\vec{\nabla}_x\right)\vec{v}\,d^3k=-n\frac{\bar{v}}{\tau}$$

$$\frac{\partial(n\bar{v})}{\partial t}-\frac{\vec{F}_{ext}}{m}n+\frac{2}{3m}\bar{E}\vec{\nabla}_x n=-n\frac{\bar{v}}{\tau}$$

As before in the text version

$$\tau\frac{\partial J_n}{\partial t}+J_n=nq\mu_n\vec{F}+\mu_n k_B T\vec{\nabla}_x n$$

ALTHOUGH BOTH THE TEXT VERSION AND MY VERSION ENDS UP WITH THE SAME ANSWER MY APPROACH IS ACCURATE SINCE IT ACCOUNTS FOR ALL THE TERMS IN THE GENERAL DRIFT-DUFFUSION EQUATION.

Drift-Diffusion Equation for Electron and Holes – And Finally

$$Taking \quad \tau \frac{\partial J_n}{\partial t} + J_n = nq\mu_n \vec{F} + \mu_n k_B T \vec{\nabla}_x n$$

If the above equation is restricted to only zero order in J_n, then $\frac{\partial J_n}{\partial t} \sim 0$

$$\therefore \quad J_n = nq\mu_n \vec{F} + \mu_n k_B T \vec{\nabla}_x n$$

$$Diffusion \quad coeff \left(Einstein \quad \mathrm{Re}lation \right) D_n = \mu_n \frac{k_B T}{q}$$

$$\therefore \quad J_n = nq\mu_n \vec{F} + qD_n \vec{\nabla}_x n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly, for holes (moves in opposite direction);

$$\therefore \quad J_p = np\mu_p \vec{F} - pD_p \vec{\nabla}_x p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Equation (1) and (2) are the Drift-Diffusion Equations for Electrons and Holes respectively

Resources

- Books
  - The Physics of Semiconductors, Kevin F. Brennan, Cambridge University Press, *New York* (1999)
  - Introduction to Modern Statistical Mechanics, David Chandler, Oxford University Press, *New York* (1987)
  - Introduction to Statistical Thermodynamics, Terrell L. Hill, Dover Publications Inc., *New York* (1986)
- Websites
  - A great site hosted by the UIUC, Some great 1-D derivations in statistical mechanics
    - http://www-ncce.ceg.uiuc.edu/tutorials/bte_dd/html/bte_dd.html
  - A good site with introductory derivations on statistical mechanics and some classical physics derivations, hosted by James Graham in UC-Berkeley
    - http://astron.berkeley.edu/~jrg/ay202/lectures.html
  - The Mathworld® site. I find it one of the most helpful to check out theorems and formulae (I checked out the divergence theorem for this derivation)
    - http://mathworld.wolfram.com/

End of Lecture