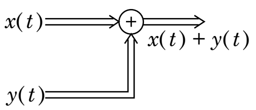
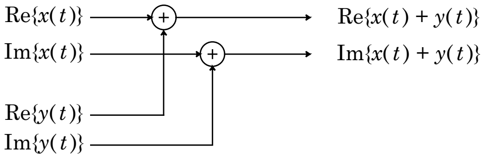
Exercise Solutions

Chapter 1

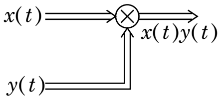
Chapter 2

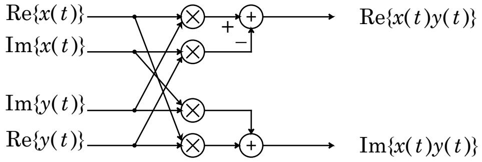
Exercise 2-1. Addition of two complex-valued signals is illustrated below:





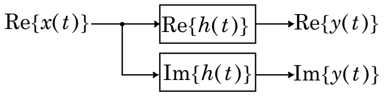
and multiplication below:

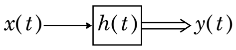




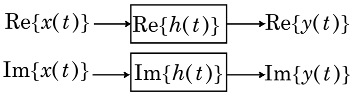
Complex addition is accomplished by two real additions, and complex multiplication by four real multiplications and two real additions.

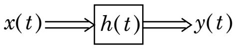
Exercise 2-2. A complex system with a real-valued input:





A real system with a complex-valued input:





Exercise 2-3. We can treat the convolution $x(t) * h(t)$ just like complex multiplication, since the convolution operation is linear — an integration. To check linearity, for a complex constant A and two input signals $x_1(t)$ and $x_2(t)$,

$$(x_1(t) + A\cdot x_2(t)) * h(t) = x_1(t) * h(t) + A\cdot(x_2(t) * h(t)) \tag{18.1}$$

following the rules of complex arithmetic. This establishes linearity.

Exercise 2-4.

$$Y(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau e^{-j2\pi ft}\, dt\ .$$

Observe that

$$e^{-j2\pi ft} = e^{-j2\pi f\tau}e^{-j2\pi f(t-\tau)} \tag{18.2}$$

so that

$$Y(f) = \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}d\tau \int_{-\infty}^{\infty} x(t-\tau)e^{-j2\pi f(t-\tau)}dt = H(f)X(f)\ , \tag{18.3}$$

after a change of variables.

Exercise 2-5. Take the Fourier transform of both sides of (2.2), getting

$$\begin{aligned}\hat{X}(f) &= \int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_m\delta(t-mT)e^{-j2\pi ft}dt \\ &= \sum_{m=-\infty}^{\infty} x_m\int_{-\infty}^{\infty}\delta(t-mT)e^{-j2\pi ft}dt \\ &= \sum_{m=-\infty}^{\infty} x_m e^{-j2\pi fmT} = X(e^{j2\pi fT})\ .\end{aligned} \tag{18.4}$$

Exercise 2-6. The impulse response of the system is $g_k = g(kT)$, and hence (2.17) gives the frequency response directly,

$$G(e^{j2\pi fT}) = \frac{1}{T}\sum_{m=-\infty}^{\infty} G(f-m/T)\ . \tag{18.5}$$

Exercise 2-7. Given $X(f) = 0$ for all $|f| > 1/(2T)$, (2.17) implies that

$$X(e^{j2\pi fT}) = \frac{1}{T}X(f) \qquad \text{for all} \quad |f| < 1/(2T)\ . \tag{18.6}$$

To get $x(t)$ from x_k, therefore, we can use, in Fig. 2-1, the following filter:

$$F(f) = \begin{cases} T; & |f|<1/(2T) \\ 0\ ; & \text{otherwise} \end{cases} \tag{18.7}$$

Exercise 2-8. From (2.24), the modulus-squared of the complex envelope $\tilde{s}(t)$ is

$$|\tilde{s}(t)|^2 = (s^2(t) + \hat{s}^2(t))/2. \tag{18.8}$$

Since the Hilbert transform is a phase-only filter, it doesn't change the energy, so the energy of $s(t)$ and its Hilbert transform $\hat{s}(t)$ are the same. Hence, the complex envelope energy is identical to that of $s(t)$.

Exercise 2-9. From (2.24), the real envelope is $e(t) = \sqrt{2}\,|\tilde{s}(t)| = (s^2(t) + \hat{s}^2(t))^{1/2}$, which is the magnitude of the phase splitter output, before the complex exponential, and is clearly independent of the carrier frequency.

Exercise 2-10. First show that $S < \infty$ implies BIBO. Suppose the input is bounded by $x_k \le L$. Then

$$|y_k| = |\sum\nolimits_{m=-\infty}^{\infty} h_m x_{k-m}| \le L \sum\nolimits_{m=-\infty}^{\infty} |h_m| = LS < \infty . \tag{18.9}$$

Then show that if $S = \infty$ there exists a bounded input such that the output is unbounded. Such an input is

$$x_k = \begin{cases} h^*_{-k} / |h_{-k}| ; & k \text{ such that } h_k \ne 0 \\ 0 ; & k \text{ such that } h_k = 0 \end{cases} \tag{18.10}$$

Exercise 2-11.

(a) If the sequence is left-sided (2.37) becomes

$$\sum\nolimits_{k=-\infty}^{K} |h_k| \cdot |z|^{-k} < \infty \quad . \tag{18.11}$$

for some K. This sum can be rewritten

$$|z|^{-K} \sum\nolimits_{k=0}^{\infty} |h_K^{-k}| \cdot |z|^k \tag{18.12}$$

and we recognize that the $|z|^{-K}$ cannot affect convergence except at $|z| = 0$ and $|z| = \infty$. All the terms in the sum are positive powers of $|z|$, and hence if they converge for some $|z| = R$ they must converge for all smaller $|z|$ (except possibly $|z| = 0$).

(b) If $K > 0$, the sequence is not anticausal, and there is a K-th order pole at $z = 0$, the ROC does not include $z = 0$. If $K = 0$, then $H(0) = h_K$, the ROC includes $z = 0$. If $K < 0$, there is a K-th order zero at $z = 0$, which is therefore included in the ROC.

Exercise 2-12. This follows directly from the observation that x_{k-l} for a fixed integer l has Z transform $z^{-l}X(z)$. Taking the Z transform of both sides of (2.41), we get the desired results.

Exercise 2-13. This is a straightforward evaluation. For example,

$$\langle y(t), \boldsymbol{x}(t) \rangle = \int_{-\infty}^{\infty} y(t)x^*(t)\, dt = (\int_{-\infty}^{\infty} x(t)y^*(t)\, dt)^* = \langle x(t), \boldsymbol{y}(t) \rangle^* . \tag{18.13}$$

Exercise 2-14. The inequality is obviously true (with equality) if $\mathbf{X} = \mathbf{0}$ or $\mathbf{Y} = \mathbf{0}$, so assume that $\mathbf{X} \ne \mathbf{0}$ and $\mathbf{Y} \ne \mathbf{0}$. Then we have the inequality

$$\begin{aligned} &0 \le \|\mathbf{X} - \alpha \cdot \mathbf{Y}\|^2 \\ &0 \le \|\mathbf{X}\|^2 - 2\mathrm{Re}\{\alpha^* \langle \mathbf{X},\mathbf{Y} \rangle\} + |\alpha|^2 \|\mathbf{Y}\|^2 \end{aligned} \tag{18.14}$$

If we let

$$\alpha = \langle \mathbf{X},\mathbf{Y} \rangle / \|\mathbf{Y}\|^2, \tag{18.15}$$

then the previous inequality becomes

$$0 \le \|\mathbf{X}\|^2 - |\langle \mathbf{X},\mathbf{Y} \rangle|^2 / \|\mathbf{Y}\|^2 , \tag{18.16}$$

from which the Schwarz inequality follows immediately.