EE 6100 Homework 6 Solutions

1. $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \\ 2 & -3 & 2 \\ 0 & 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y_1 = [0 \quad 1 \quad 0]\, x = C_1 x$$
$$y_2 = [1 \quad -1 \quad 0]\, x = C_2 x$$

$$M = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 2 \\ 1 & -1 & -1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 1/2 & 0 & -1/2 \\ -1/4 & 1/4 & -1/4 \end{bmatrix}$$

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \quad \tilde{B} = M^{-1}B = \begin{bmatrix} 3/4 \\ 0 \\ -1/4 \end{bmatrix}$$ needs to be nonzero for controllability

$$\tilde{C}_1 = [2 \quad 0 \quad 2]$$ needs to be nonzero for observability

$$\tilde{C}_2 = [-1 \quad 1 \quad -3]$$

The system is neither controllable nor observable with C_1. It is observable with C_2.

11.27 (a) $P = \begin{bmatrix} 1 & -4 \\ 1 & 4 \end{bmatrix}$, $|P| = 8$ so controllable

(b) This is in Jordan form ($\lambda = 0$). The bottom row of B corresponding to the Jordan block is not zero $\Rightarrow$ controllable

(c) $P = \begin{bmatrix} 0 & 6 & 48 \\ 0 & 2 & 16 \\ 1 & 4 & 32 \end{bmatrix}$, $|P| = 0$ not controllable

11.28 (a) This is partially decoupled, so go ahead and finish putting it into Jordan form.
Since the system is decoupled, you can put the individual subsystems into Jordan form & then consider the whole system.

$$A = \left[\begin{array}{c|cc|c|cc} -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right], \quad B = \left[\begin{array}{cc} 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \\ \hline 0 & 1 \\ \hline 0 & 0 \\ 1 & 0 \end{array}\right]$$

$$C = \left[\begin{array}{c|cc|c|cc} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\left| \lambda I - \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \right| = \begin{vmatrix} \lambda+2 & -1 \\ 0 & \lambda+1 \end{vmatrix} = \lambda^2 + 3\lambda + 2$$

corresponding eigenvectors are:

$$\lambda = -2: \quad \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} m_1 = 0 \Rightarrow m_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda = -1: \quad \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} m_2 = 0 \Rightarrow m_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$M^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} M = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$$

so the transformed system is $\left\{ \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \right\}$

The other system to be decoupled is

$$\left\{ \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

$$\lambda = -3: \quad \begin{bmatrix} \lambda+3 & -1 \\ 0 & \lambda+1 \end{bmatrix} m_1 = \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix} m_1 = 0 \Rightarrow m_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda = -1: \quad \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} m_2 = 0 \Rightarrow m_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \frac{1}{2}$$

$$M^{-1}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} = \begin{bmatrix}-\frac{1}{2} & 0\\ \frac{1}{2} & 0\end{bmatrix}; \quad \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} M = \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$$

The transformed system is:

$$\left\{\begin{bmatrix}-3 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}-\frac{1}{2} & 0\\ \frac{1}{2} & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\right\}$$

Put this all together:

$$\tilde{A} = \begin{bmatrix}-1 & & & & & \\ 0 & -2 & 0 & & 0 & \\ 0 & 0 & -1 & & & \\ 0 & & & -3 & & \\ 0 & 0 & & & -3 & 0\\ 0 & & & & 0 & -1\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\\ 0 & 1\\ -\frac{1}{2} & 0\\ \frac{1}{2} & 0\end{bmatrix}$$

There are 3 blocks associated with $\lambda = -1$, the corresponding rows of B are not linearly independent $\Rightarrow$ not controllable

$$\tilde{C} = \begin{bmatrix}1 & 2 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 1\end{bmatrix}$$

The columns of C corresponding to the -1 blocks are not linearly independent $\Rightarrow$ not observable

I could have seen these results just by examining the original system. The -1 eigenvalues are in 3 different blocks. With 2 inputs + 2 outputs, there is no way to get 3 linearly ind. col of C or rows of B assoc. with this eigenvalue

(b) $P=\begin{bmatrix}1 & 0 & -6\\ 6 & -6 & -6\\ 5 & -6 & 0\end{bmatrix}$, $|P| = -36 - 6(-6) = 0$

not controllable

$Q=\begin{bmatrix}1 & 0 & 0\\ -6 & 1 & 0\\ 25 & -6 & 1\end{bmatrix}$, $|Q| = 1$ observable

(c) partially decoupled. Modes associated with $\lambda=-5$ have linearly indep rows of B & col of C associated with them. Examine upper left block & associated subsystem

$$\left\{\begin{bmatrix}-1 & 3\\ -3 & -1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}-3 & -4\\ 4 & -3\end{bmatrix}\frac{1}{25}\right\}$$

$\lambda = -1 \pm 3j$, So this block can truly be decoupled from the other blocks since the eigenvalues are not -5. Consider observability & controllability of this block.

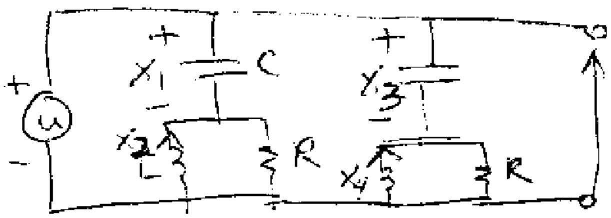
$P=\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & -3 & 0\end{bmatrix}$, rank$(P) = 2 \Rightarrow$ controllable

$Q=\begin{bmatrix}\begin{bmatrix}-3 & -4\\ 4 & -3\end{bmatrix}\frac{1}{25}\\ \begin{matrix}15 & -5\\ +5 & 15\end{matrix}\end{bmatrix}$, rank$(Q) = 2 \Rightarrow$ observable

The subsystems are all controllable + observable & the blocks that have repeated eigenvalues have lin. indep rows of B + col of C assoc. with them $\Rightarrow$ The system is obs + cont.

11.32 (a) The two circuits are in parallel + are identical. The circuits cannot be controlled separately (so the individual states cannot be transferred to arbitrary states). The output is equal to the input, $y = u$ so $C = 0$. The system is not observable.

You can also show the system is not controllable mathematically.



KVL: $u = x_1 + R(i_c - x_2)$

$$u = x_1 + R(C\dot{x}_1 - x_2)$$

$$\Rightarrow \dot{x}_1 = \frac{1}{RC}\left[-x_1 + R x_2 + \boxed{u}\right]$$

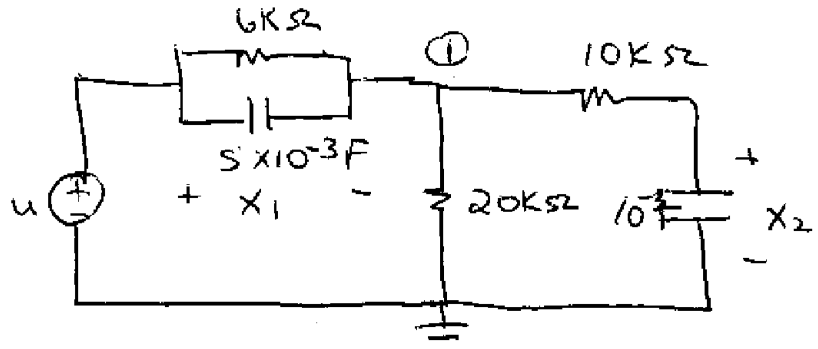
$$L\dot{x}_2 = u - x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1/RC & \frac{1}{C} \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/RC \\ 1/L \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1/RC & 1/C & 0 & 0 \\ -1/L & 0 & 0 & 0 \\ 0 & 0 & -1/RC & 1/C \\ 0 & 0 & -1/L & 0 \end{bmatrix} x + \begin{bmatrix} 1/RC \\ 1/L \\ 1/RC \\ 1/L \end{bmatrix} u$$

form P + you will see that $|P| = 0$

(b)



KCL at ①: $\frac{-x_1}{6000} - 5\times10^{-3}\dot{x}_1 + \frac{-x_1+u}{20000} + \frac{u-x_1-x_2}{10000} = 0$

$$5\times10^{-3}\dot{x}_1 = \frac{-19}{60000}x_1 - \frac{x_2}{10000} + \frac{3}{20000}u$$

$$\dot{x}_1 = -\frac{19}{300}x_1 - \frac{x_2}{50} + \frac{3}{100}u$$

$$10^{-3}\dot{x}_2 = \frac{u-x_1-x_2}{10000} \longrightarrow \dot{x}_2 = -.01x_1 - .01x_2 + .01u$$

$$\dot{x} = \begin{bmatrix} -19/300 & -1/50 \\ -.1 & -.1 \end{bmatrix} x + \begin{bmatrix} 3/100 \\ .1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

$P = \begin{bmatrix} 3/100 & -39/10000 \\ 0.1 & -13/1000 \end{bmatrix}$, ~~not~~ $|P| = 0$ so not controllable

$Q = \begin{bmatrix} 0 & 1 \\ -.1 & -.1 \end{bmatrix}$, $|Q| \neq 0$ so observable

11.33 $G'(t_1, t_0) = \int_{t_0}^{t_1} \Phi(t_0, t) B(t) B^T(t) \Phi^T(t_0, t)\, dt$

If A & B real, the controllability gramian is

$$G(t_1, t_0) = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau)\, B^T(\tau) \Phi^T(t_1, \tau) d\tau$$

prove $G(t_1, t_0) > 0$ iff $G'(t_1, t_0)$

Note: if a matrix $P > 0$ then $x^T P x > 0$ for all vectors x. Let $x = Tz$ where T is invertible. Then $x^T P x = z^T T^T P T z > 0$ for all vectors z. Therefore,

$P > 0 \iff T^T P T > 0$ for all invertible matrices T.

similarly $P > 0 \iff T P T^T > 0$

Consider $\Phi(t_1, t_0)\, G'(t_1, t_0) \Phi^T(t_1, t_0)$

This is positive def. iff $G'(t_1, t_0) > 0$, so prove that this is equal to $G(t_1, t_0) > 0$

$\Phi(t_1, t_0) G'(t_1, t_0) \Phi^T(t_1, t_0)$

$= \int_{t_0}^{t_1} \Phi(t_1, t) B(t) B^T(t) \Phi^T(t_0, t) \Phi^T(t_1, t_0)\, dt$

note: $\Phi^T(t_0, t) \Phi^T(t_1, t_0) = \left[\Phi(t_1, t_0) \Phi(t_0, t)\right]^T$
$= \Phi(t_1, t)^T$

$= G(t_1, t_0)$