1. a) $\dot{y} + 4y = 3x$, $x = \sin(2t)$, $y(0) = 1$

$sY(s) - y(0) + 4Y(s) = 3X(s)$

$(s+4)Y(s) = \dfrac{6}{s^2+4} + 1$

$Y(s) = \dfrac{6}{(s^2+4)(s+4)} + \dfrac{1}{s+4}$

$= \dfrac{c_1 s + c_2}{s^2+4} + \dfrac{c_3}{s+4} + \dfrac{1}{s+4}$

$c_3 = \dfrac{3}{10}$

$6 = (c_1 s + c_2)(s+4) + \dfrac{3}{10}(s^2+4) \Rightarrow c_1 = -\dfrac{3}{10}$

$6 = 4c_2 + \dfrac{6}{5}$

$\dfrac{4}{5} = c_2$

$Y(s) = \dfrac{-\frac{3}{10}s + \frac{4}{5}}{s^2+4} + \dfrac{13/10}{s+4}$

$= \dfrac{-\frac{3}{10}s}{s^2+4} + \dfrac{6}{10}\dfrac{2}{s^2+4} + \dfrac{13/10}{s+4}$

$y(t) = \left(-\dfrac{3}{10}\cos(2t) + \dfrac{4}{10}\sin(2t) + \dfrac{13}{10}e^{-4t}\right)u(t)$

b) $\ddot{y} + 4\dot{y} + 20y = 2\dot{x} - x$, $x(t) = u(t)$, $y(0) = 0, \dot{y}(0) = 1$

$s^2 Y(s) - sy(0) - \dot{y}(0) + 4sY(s) - 4y(0) + 20Y(s)$

$$= 2sX(s) - 2x(0)^0 - X(s)$$

$(s^2 + 4s + 20) Y(s) = 1 + 2 - \frac{1}{5}$

$$Y(s) = \frac{3}{(s+2)^2 + 16} - \frac{1}{\left((s+2)^2 + 16\right)s}$$

$$= \frac{3}{(s+2)^2 + 16} + \frac{c_1 s + c_2}{(s+2)^2 + 16} + \frac{c_3}{s}$$

$$c_3 = -\frac{1}{20}, \quad c_1 = \frac{1}{20}, \quad c_2 = \frac{1}{5}$$

$$Y(s) = \frac{3}{4} \cdot \frac{4}{(s+2)^2 + 16} + \underbrace{\frac{\frac{1}{20} s + \frac{1}{5}}{(s+2)^2 + 16}}_{\frac{1}{20}\frac{(s+2)}{(s+2)^2 + 16} + \frac{1}{10}\frac{1}{(s+2)^2 + 16}} - \frac{\frac{1}{20}}{s}$$

$$y(t) = \frac{3}{4} e^{-2t} \sin(4t) + \frac{1}{20} e^{-2t} \cos(4t) + \frac{1}{40} e^{-2t} \sin(4t)$$

$$\text{for } t \geq 0$$

c) $\ddot{y} + 7\dot{y} + 12y = 6x, \quad x = u(t), \quad y(0) = 0 \quad \dot{y}(0) = -2$

$$s^2 Y(s) - s y(0) - \dot{y}(0) + 7(s Y(s) - y(0)) + 12 Y(s) = \frac{6}{s}$$

$$(s^2 + 7s + 12) Y(s) = \frac{6}{s} - 2$$

$$Y(s) = \frac{-2s + 6}{s(s^2 + 7s + 12)} = \frac{C_1}{s} + \frac{C_2}{s+4} + \frac{C_3}{s+3}$$

$$C_1 = \frac{1}{2}, \quad C_2 = \frac{-2s+6}{s(s+3)}\bigg|_{s=-4} = -\frac{1}{2}$$

$$C_3 = \frac{-2s+6}{s(s+4)}\bigg|_{s=-3} = -4$$

$$Y(s) = \frac{1/2}{s} - \frac{1/2}{s+4} - \frac{4}{s+3}$$

$$y(t) = \frac{1}{2} - \frac{1}{2}e^{-4t} - 4e^{-3t}, \quad t \geq 0$$

d)

$$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 20y = x(t), \quad y(0) = 1, \dot{y}(0) = -2, \ x(t) = 2u(t)$$

$$s^2 Y(s) - s y(0) - \dot{y}(0) + 9(s Y(s) - y(0)) + 20 Y(s)$$

$$= X(s)$$

$$(s^2 + 9s + 20) Y(s) = s - 2 + 9 + \frac{2}{s}$$

$$Y(s) = \frac{s^2 + 7s + 2}{s(s^2 + 9s + 20)} = \frac{s^2 + 7s + 2}{s(s+4)(s+5)}$$

$$Y(s) = \frac{c_1}{s} + \frac{c_2}{s+4} + \frac{c_3}{s+5}$$

$$= \frac{1/10}{s} + \frac{5/2}{s+4} + \frac{-8/5}{s+5}$$

$$y(t) = \frac{1}{10} + \frac{5}{2}e^{-4t} - \frac{8}{5}e^{-5t}, \quad t \geq 0$$