Where We Are in J&D

• Material from:
  – Sec. 4.9.2 on spatial averaging
  – Sec. 3.3.4 on co-arrays
  • We will use uniform weights
  – Notes by Doug Williams
Why Spatial Averaging?

- Last lecture, we looked at temporal averaging

- Sometimes the source is moving too fast to employ a lot of temporal averaging

- If the SNR is still too low, it helps to have some redundancy in the array
Ex: Uniform Linear Array

• For a ULA, with multiple incoherent sources, and no noise:

\[
R_y = \begin{bmatrix}
R_0 & R_1 & R_2 & \cdots & R_{M-1} \\
R_1^* & R_0 & R_1 & \cdots & R_{M-2} \\
R_2^* & R_1^* & R_0 & \cdots & R_{M-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
R_{M-1}^* & R_{M-2}^* & R_{M-3}^* & \cdots & R_0
\end{bmatrix}
\]

← Toeplitz structure
Forcing a Toeplitz Structure (1)

\[ \hat{R}_y = \frac{1}{L} \sum_{l=0}^{L-1} y(l)y^H(l) \]

\[ \hat{R}_y = \begin{bmatrix}
\hat{R}_{0,0} & \hat{R}_{0,1} & \hat{R}_{0,2} & \cdots & \hat{R}_{0,M-1} \\
\hat{R}^*_{0,1} & \hat{R}_{1,1} & \hat{R}_{1,2} & \cdots & \hat{R}_{1,M-1} \\
\hat{R}^*_{0,2} & \hat{R}^*_{1,2} & \hat{R}_{2,2} & \cdots & \hat{R}_{2,M-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\hat{R}^*_{0,M-1} & \hat{R}^*_{1,M-1} & \hat{R}^*_{2,M-1} & \cdots & \hat{R}_{M-1,M-1} \\
\end{bmatrix} \]

- One idea: Average along diagonals
- Elements near main diagonal get larger variability reduction

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Forcing a Toeplitz Structure (2)

- Now we’re guaranteed a Toeplitz structure:

\[
\hat{R}_y = \begin{bmatrix}
\hat{R}_0 & \hat{R}_1 & \hat{R}_2 & \ldots & \hat{R}_{M-1} \\
\hat{R}^*_1 & \hat{R}_0 & \hat{R}_1 & \ldots & \hat{R}_{M-2} \\
\hat{R}^*_2 & \hat{R}^*_1 & \hat{R}_0 & \ldots & \hat{R}_{M-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\hat{R}^*_{M-1} & \hat{R}^*_{M-2} & \hat{R}^*_{M-3} & \ldots & \hat{R}_0
\end{bmatrix}
\]

- …but we’re not guaranteed a nonnegative definite one!
  - i.e., at least one eigenvalue may become negative
What Can Go Wrong

• Resulting matrix not guaranteed to be nonnegative definite!

• Also, procedure can introduce some bias into angle estimates
  – Usually only a major problem if coherent signals are present

• One solution: maximum-likelihood structured covariance estimation
  – Expectation-Maximization (EM) algorithm (computationally intensive)
Subaperture Concept

• Suppose we have a 5 element ULA:

  ● ● ● ● ● ●

• Form 3 subarrays:

\[
\hat{R}_y = \begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
\]

\[
\frac{1}{3} \sum = \hat{R}_{sub} = \begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
\]
Subaperture Averaging: Pros and Cons

• Advantages
  – All elements of estimated spatial correlation matrix (SCM) get the same improvement
  – Resulting SCM estimate is guaranteed to be nonnegative definite
  – Helps reduce problems with coherent sources
    • Sinusoidal components along diagonals get smoothed out

• Disadvantage
  – Individual subapertures have lower resolution
The Tradeoff

- Lower variability in covariance estimate comes at the expense of lower resolution

- 4 small subarrays (bad resolution, low/good variability):

- 2 big subarrays (good resolution, high/bad variability)
Forward/Backward Averaging

- From Prob. 7.11 on p. 418 of J&D
- Consider the data for a uniform linear array:
  \[ y = [y_0 \ y_1 \ \cdots \ y_{M-2} \ y_{M-1}]^T \]
- Define:
  \[ y_r^* = [y_{M-1}^* \ y_{M-2}^* \ \cdots \ y_1^* \ y_0^*]^T \]
- Describes same signal and noise situation, but coherence effects differ

\[
\hat{R}_{fb} = \frac{1}{2L} \sum_{l=0}^{L-1} \left\{ y(l)y^H(l) + y_r^*(l)[y_r^*(l)]^H \right\}
\]
General Subarrays

- 12 element array
- 4, 3-element subarrays
- 2, 6-element subarrays
Interpreting the SCM

- SCM can be viewed as samples of the **correlation function** for the entire field
- Samples are taken at **differences** between sensor locations
- For a single plane wave:
  \[
  \left[ R \right]_{m_1,m_2} = P_s \exp\{ jk^0 \cdot (\vec{x}_{m_1} - \vec{x}_{m_2}) \} \]

- For a general WSS field:
  \[
  \left[ R \right]_{m_1,m_2} = R_f (\vec{x}_{m_1} - \vec{x}_{m_2})
  \]
  where 
  \[
  R_f (\chi) = E[ f(0) f(\chi) ]
  \]
The Coarray

• Set of differences between all pairs of sensor locations is called the coarray

\[ \bigcup_{m_1,m_2} \{ \vec{x}_{m_1} - \vec{x}_{m_2} \} \]

• Differences are known as lags

\[ \vec{z}_{m_1,m_2} \equiv \vec{x}_{m_1} - \vec{x}_{m_2} \]

• Elements of the coarray have associated coarray values, which is the number of distinct baselines (pairs of actual sensors) with the same vector difference
Properties of Co-arrays

• Dimension of coarray is same as dimension of array
  – Linear <-> Linear
  – Planar <-> Planar

• Certain lags must exist
  \[ \vec{Z}_{m,m} = 0 \] (coarray value of \( M \))
  \[ \vec{Z}_{m_1,m_2} = -\vec{Z}_{m_2,m_1} \]
Redundancies

- Repeated lags (i.e. co-array values > 1) are called **redundancies**
- Zero-lag redundancies occur along main diagonal of SCM
  - Ideally equal if only incoherent signals are present
  - Usually not equal due to noise or coherent signals
- Minimum no. of distinct positive lags: $M$
- Maximum no. of distinct positive lags: $M(M-1)/2$ (not achievable if $M>4$)
Linear Co-array Examples

- 4-sensor uniformly spaced:
  - \( d \) \( d \) \( d \) \( d \) \( \cdot \cdot \cdot \)

- 4-sensor “perfect” array:
  - \( d \) \( 3d \) \( 2d \) \( \cdot \cdot \cdot \)

Mirrored