Spatial Averaging and Co-arrays

ECE 6279: Spatial Array Processing
Fall 2013
Lecture 13

Prof. Aaron D. Lanterman
School of Electrical & Computer Engineering
Georgia Institute of Technology
AL: 404-385-2548
<lanterma@ece.gatech.edu>

Why Spatial Averaging?

• Last lecture, we looked at temporal averaging

• Sometimes the source is moving too fast to employ a lot of temporal averaging

• If the SNR is still too low, it helps to have some redundancy in the array

Where We Are in J&D

• Material from:
  – Sec. 4.9.2 on spatial averaging
  – Sec. 3.3.4 on co-arrays
    • We will use uniform weights
  – Notes by Doug Williams

Ex: Uniform Linear Array

• For a ULA, with multiple incoherent sources, and no noise:

$$R_y = \begin{bmatrix}
R_0 & R_1 & R_2 & \cdots & R_{M-1} \\
R_1^* & R_0 & R_1 & \cdots & R_{M-2} \\
R_2^* & R_1^* & R_0 & \cdots & R_{M-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
R_{M-1}^* & R_{M-2}^* & R_{M-3}^* & \cdots & R_0
\end{bmatrix}$$

→ Toeplitz structure
Forcing a Toeplitz Structure (1)

\[ \hat{R}_y = \frac{1}{L} \sum_{i=0}^{L-1} y(l)y^H(l) \]

- One idea: Average along diagonals
- Elements near main diagonal get larger variability reduction

Forcing a Toeplitz Structure (2)

\[ \hat{R}_y = \begin{bmatrix} \hat{R}_0 & \hat{R}^* & \hat{R} & \cdots & \hat{R}_{M-1} \\ \hat{R}^* & \hat{R}_0 & \hat{R} & \cdots & \hat{R}_{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{R} & \cdots & \cdots & \cdots & \hat{R}_0 \end{bmatrix} \]

- Now we’re guaranteed a Toeplitz structure:
- ...but we’re not guaranteed a nonnegative definite one!
  - i.e., at least one eigenvalue may become negative

What Can Go Wrong

- Resulting matrix not guaranteed to be nonnegative definite!
- Also, procedure can introduce some bias into angle estimates
  - Usually only a major problem if coherent signals are present
- One solution: maximum-likelihood structured covariance estimation
  - Expectation-Maximization (EM) algorithm (computationally intensive)

Subaperture Concept

- Suppose we have a 5 element ULA:
  - Form 3 subarrays:
    - [Diagram of subaperture concept]

\[ \hat{R}_y = \begin{bmatrix} \hat{R}_0 & \hat{R}^* & \hat{R} & \cdots & \hat{R}_{M-1} \\ \hat{R}^* & \hat{R}_0 & \hat{R} & \cdots & \hat{R}_{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{R} & \cdots & \cdots & \cdots & \hat{R}_0 \end{bmatrix} \]
Subaperture Averaging: Pros and Cons

• **Advantages**
  – All elements of estimated spatial correlation matrix (SCM) get the same improvement
  – Resulting SCM estimate is guaranteed to be nonnegative definite
  – Helps reduce problems with coherent sources
    • Sinusoidal components along diagonals get smoothed out

• **Disadvantage**
  – Individual subapertures have lower resolution

The Tradeoff

• Lower variability in covariance estimate comes at the expense of lower resolution

• 4 small subarrays (bad resolution, low/good variability):

• 2 big subarrays (good resolution, high/bad variability)

Forward/Backward Averaging

• From Prob. 7.11 on p. 418 of J&D
• Consider the data for a uniform linear array:
  \[ y = \begin{bmatrix} y_0 & y_1 & \cdots & y_{M-2} & y_{M-1} \end{bmatrix}^T \]

• Define:
  \[ y_r^* = \begin{bmatrix} y_{M-1}^* & y_{M-2}^* & \cdots & y_1^* & y_0^* \end{bmatrix}^T \]

• Describes same signal and noise situation, but coherence effects differ

\[ \hat{R}_{fb} = \frac{1}{2L} \sum_{l=0}^{L-1} \{ y(l)y^H(l) + y_r^*(l)[y_r^*(l)]^H \} \]

General Subarrays

• 12 element array
• 4, 3-element subarrays

• 2, 6-element subarrays
Interpreting the SCM

- SCM can be viewed as samples of the correlation function for the entire field
- Samples are taken at differences between sensor locations
- For a single plane wave:
  \[
  [R]_{m_1,m_2} = P_s \exp\{jk \cdot (\vec{x}_{m_1} - \vec{x}_{m_2})\}
  \]
- For a general WSS field:
  \[
  [R]_{m_1,m_2} = R_f(\vec{x}_{m_1} - \vec{x}_{m_2})
  \]
  where \( R_f(\chi) = E[f(0)f(\chi)] \)

The Coarray

- Set of differences between all pairs of sensor locations is called the coarray
  \[
  \bigcup_{m_1,m_2} \{\vec{x}_{m_1} - \vec{x}_{m_2}\}
  \]
- Differences are known as lags
- Elements of the coarray have associated coarray values, which is the number of distinct baselines (pairs of actual sensors) with the same vector difference

Properties of Co-arrays

- Dimension of coarray is same as dimension of array
  - Linear <-> Linear
  - Planar <-> Planar
- Certain lags must exist
  \[
  \vec{z}_{m,m} = 0 \quad \text{(coarray value of } M) \]
  \[
  \vec{z}_{m_1,m_2} = -\vec{z}_{m_2,m_1}
  \]

Redundancies

- Repeated lags (i.e. co-array values > 1) are called redundancies
- Zero-lag redundancies occur along main diagonal of SCM
  - Ideally equal if only incoherent signals are present
  - Usually not equal due to noise or coherent signals
- Minimum no. of distinct positive lags: \( M \)
- Maximum no. of distinct positive lags: \( M(M-1)/2 \) (not achievable if \( M > 4 \))
Linear Co-array Examples

- 4-sensor uniformly spaced:
  \[ d \quad d \quad d \]
  \[-3d \quad -2d \quad -d \]
  \[ d \quad 2d \quad 3d \]

- 4-sensor “perfect” array:
  \[ d \quad 3d \quad 2d \]