Sampling Theorems for Uniform and Periodic Nonuniform MIMO Sampling of Multiband Signals

Raman Venkataramani and Yoram Bresler*

Abstract—**We examine a multiple-input multiple-output (MIMO) sampling scheme for a linear time-invariant continuous-time MIMO channel. The input signals are modeled as multiband signals with different spectral supports, and the channel outputs are sampled on either uniform or periodic nonuniform sampling sets, with possibly different but commensurate intervals on the different outputs. This scheme encompasses Papoulis' generalized sampling and several nonuniform sampling schemes as special cases. We derive necessary and sufficient conditions on the channel and the sampling rate that allow stable perfect reconstruction of the inputs, or equivalently, perfect inversion of the channel. We argue that, from an implementation viewpoint, it is desirable that the reconstruction filters have continuous frequency responses, and derive necessary and sufficient conditions that guarantee this continuity property. The frequency responses of the reconstruction filters are specified as solutions to a system of linear equations. Finally, we demonstrate that perfect reconstruction of the inputs is possible even when the channel outputs are sampled at a total combined rate that is smaller than the some of the individual Nyquist rates for the inputs.**

Keywords: stable sampling, sub-Nyquist sampling, interpolation, reconstruction, multiple-input-multiple-output channel, multichannel deconvolution, MIMO equalization, source separation, multiband signals.

I. Introduction

The study of multiple-input multiple-output (MIMO) channel equalization is motivated by applications in multichannel deconvolution and multiple source separation. Some example applications where MIMO channels arise are multiuser or multiaccess wireless communications and space-time coding with antenna arrays or telephone digital subscriber loops [1–4], multisensor biomedical signals [5, 6], multi-track magnetic recording [7], multiple speaker (or other acoustic source) separation with microphone arrays [8, 9], geophysical data processing [10], and multichannel image restoration [11, 12].

In practice, the MIMO channel equalizer is implemented using digital signal processors. However, the channel inputs and outputs are continuous-time signals, implying that the channel outputs need to be sampled prior to processing by the digital system. Hence the problem is equivalent to reconstructing the channel inputs from the sampled output signals. In other words, the MIMO channel inversion problem can be restated as one in sampling theory, and we call this sampling scheme *MIMO sampling*. To focus on the sampling issues, we restrict our attention in this paper to the scenario of a linear time-invariant continuous-time MIMO channel with known frequency response matrix. The harder problem of sampling conditions for blind channel inversion is left for future work.

The study of MIMO sampling has useful practical implications. Most work to date on multichannel deconvolution has addressed discrete-time channel models, apparently assuming that each output is sampled at the appropriate common Nyquist rate sufficient for reconstruction of each output. However as we demonstrate in this paper, this is not necessary, and appropriately chosen uniform or nonuniform sampling schemes with lower average sampling density suffice for perfect reconstruction of the MIMO channel inputs.

Although motivated by real world problems, MIMO sampling is an important problem in sampling theory in its own right. Several sampling schemes can be expressed as special cases of the MIMO setting. For example, for a single-input multiple-output (SIMO) channel, the outputs are filtered and uniformly sampled versions of a single input signal. In other words, this is precisely Papoulis' generalized sampling [13]. Additionally, if the channel filters are chosen to be pure delays, one obtains multicoset or periodic nonuniform sampling of the input which has been widely studied [14–25], as it allows us to approach the Landau minimum sampling for multiband signals [26]. Seidner and Feder [27] provide a natural generalization of Papoulis' sampling expansions for a vector inputs whose components are bandlimited to $[-B, B]$. Their sampling scheme is clearly a special case of MIMO sampling. We deal only with multiband signal spaces, and we refer the reader to [28] for some results on multichannel sampling for general signal spaces such as wavelet and spline spaces.

Figure 1 is the block diagram for MIMO sampling. The channel is shown to the left of the dashed line, and its inputs $x_r(t)$ are assumed to be continuous-time signals. The channel is modeled as a linear time-invariant system. The channel outputs are sampled at a uniform rate of $1/T$ to produce discrete-time sequences, $z_p[n]$. From a practical viewpoint, we can interpret this as the sampling step prior to processing digitally. The reconstruction block, shown to the right of the dashed line, inverts the MIMO channel to produce estimates $\tilde{x}_r(t)$ of the input signals.

As appropriate in many applications, we assume the input signals are multiband, with possibly different band structure (spectral support) for the different inputs. Figure 2 shows such an example for a two-input MIMO channel, which will be used throughout the paper for illustrative purposes.

The problem that we address in this paper is a special, uniform sampling case of the general MIMO sampling problem introduced in [29, 30]. We study the following issues in this paper: (a) the relation of stable MIMO sampling to frame theory; and (b) the necessary and sufficient conditions on the channel allowing to achieve perfect reconstruction of the inputs under uniform sampling. Even though we consider only *uniform* sampling of the MIMO channel outputs, we shall see later that this sampling scheme is fairly general, and it encompasses most periodic nonuniform sampling of the channel outputs, with sampling at different rates on different channels.

We derived necessary sampling density conditions for the general MIMO sampling problem in [29, 30]. We showed that stable sampling and reconstruction of the inputs imposes lower bounds on the sampling densities on the various channels, regardless of whether the sampling is uniform or not. These results are analogues of Landau's classic minimum density results for multiband single-channel sampling [26]. It is not clear whether those conditions are sufficient, however they indicate the potential for reduction in the sampling density needed for stable sampling, relative to the Nyquist rate. In this paper, we demonstrate how to achieve stable sampling and reconstruction at rates lower than the Nyquist rate. We can think of these results as partial sufficient conditions for stable MIMO sampling, although we do not provide explicit bounds on the sampling densities themselves. These results thus complement our results in [29].

This paper is organized as follows. Section II formulates the problems and introduces some notation and definitions used in the rest of the paper. In Section III we present models for the channel and reconstruction, demonstrating that various nonuniform sampling schemes can be reduced to uniform sampling of the outputs of a modified channel. Section IV deals with the problem of perfect reconstruction of the channel inputs. We explore the connection between MIMO sampling and frame

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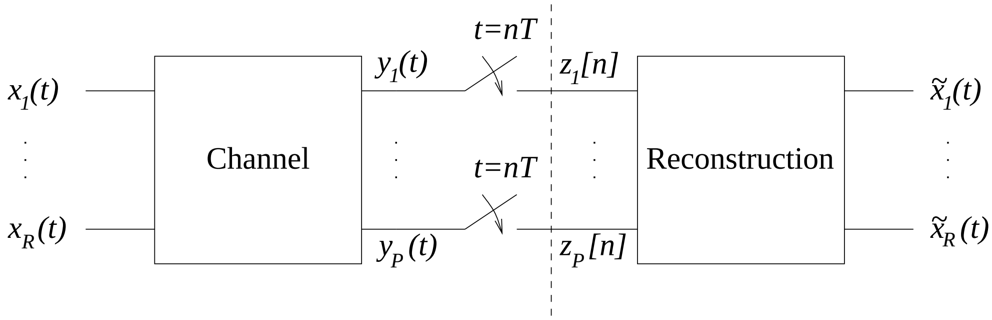

Fig. 1. The MIMO sampling problem.

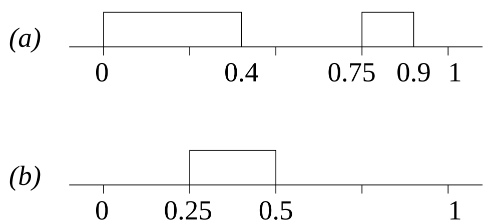

Fig. 2. Example multiband input spectra $X_0(f)$ and $X_1(f)$ to a two-input MIMO channel.

theory. The computation of the frame bounds enables us to determine necessary conditions on the input signal spaces, the channel characteristics, and the sampling rate for the existence of reconstruction filters that achieves stable and perfect reconstruction of the inputs. We also present additional conditions under which there exist reconstruction filters that are continuous in the frequency domain. This, as elaborated later, is important from the viewpoint of finite impulse response (FIR) filter design.

II. Definitions and Notation

We denote the Fourier transform of a continuous-time square-integrable signal $x(t)$ by

$$X(f) = \int_{\mathbb{R}} x(t) e^{-j2\pi f t} dt.$$

Similarly, for a discrete-time signal $y[n]$, we define its Fourier transform be

$$Y[\nu] = \sum_{n \in \mathbb{Z}} y[n] e^{-j2\pi\nu n}.$$

In general, we denote discrete-time and continuous-time signals (either scalar-valued or vector-valued) using lower-case letters, and their Fourier transforms by the corresponding upper-case letters. Let the class of continuous, finite-energy signals bandlimited to the set of frequencies $\mathcal{F}$ be

$$\mathcal{B}(\mathcal{F}) = \{x \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : X(f) = 0, \, \forall f \notin \mathcal{F}\}. \qquad (1)$$

Denote the interior and closure of a set $\mathcal{S} \subseteq \mathbb{R}$ by $\operatorname{int} \mathcal{S}$ and $\overline{\mathcal{S}}$ respectively, the class of complex-valued matrices of size $M \times N$ by $\mathbb{C}^{M \times N}$, the conjugate-transpose of $\boldsymbol{A}$ by $\boldsymbol{A}^H$, and its pseudo inverse by $\boldsymbol{A}^\dagger$. For a given matrix $\boldsymbol{A}$, let $\boldsymbol{A}_{\mathcal{R},\mathcal{C}}$ denotes the submatrix of $\boldsymbol{A}$ corresponding to rows indexed by the set $\mathcal{R}$ and columns by the set $\mathcal{C}$. The quantity $\boldsymbol{A}_{\bullet,\mathcal{C}}$ denotes a submatrix formed by keeping all rows of $\boldsymbol{A}$, but only columns indexed by $\mathcal{C}$, while $\boldsymbol{A}_{\mathcal{R},\bullet}$ denotes the submatrix formed by retaining rows indexed by $\mathcal{R}$ and all columns. We use a similar notation for vectors. Hence $\boldsymbol{X}_{\mathcal{R}}$ is the subvector of $\boldsymbol{X}$ corresponding to rows indexed by $\mathcal{R}$. We always apply the subscripts of a matrix before the superscript. So $\boldsymbol{A}^H_{\mathcal{R},\mathcal{C}}$ is the conjugate-transpose of $\boldsymbol{A}_{\mathcal{R},\mathcal{C}}$. When dealing with singleton index sets: $\mathcal{R} = \{r\}$ or $\mathcal{C} = \{c\}$, we omit the curly braces for readability. Therefore $\boldsymbol{A}_{r,\bullet}$ and $\boldsymbol{A}_{\bullet,c}$ are the r-th row and the c-th column of $\boldsymbol{A}$ respectively. For convenience, we always number the rows and columns of a finite-size matrix starting from 0. For infinite-size matrices, the row and column indices range over $\mathbb{Z}$.

The identity matrix of size $N \times N$ is denoted by $\boldsymbol{I}_N$, and the zero matrix, by $\boldsymbol{0}$. We denote the indicator function by $\chi(\cdot)$. Finally suppose that $\mathcal{S}$ is a subset of $\mathbb{R}$ or $\mathbb{Z}$, and a is an element of $\mathbb{R}$ or $\mathbb{Z}$. Then

$$\begin{aligned} \mathcal{S} \oplus a &= \{s + a : s \in \mathcal{S}\}, \\ \mathcal{S} \ominus a &= \{s - a : s \in \mathcal{S}\}, \\ a\mathcal{S} &= \{as : s \in \mathcal{S}\}, \\ \mathcal{S} \bmod a &= \{s \bmod a : s \in \mathcal{S}\}. \end{aligned}$$

denote respectively the positive and negative translations, scaling, and the modulus of $\mathcal{S}$ by a.

III. Sampling and Reconstruction Models

Let the input and output signals of the MIMO channel depicted in Figure 1 be represented in vector form as

$$\boldsymbol{x}(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ \vdots \\ x_{R-1}(t) \end{pmatrix} \quad \text{and} \quad \boldsymbol{y}(t) = \begin{pmatrix} y_0(t) \\ y_1(t) \\ \vdots \\ y_{P-1}(t) \end{pmatrix}. \qquad (2)$$

For convenience, define $\mathcal{R} = \{0, 1, \ldots, R-1\}$ and $\mathcal{P} = \{0, 1, \ldots, P-1\}$. These sets index the components of the input and the output vectors. For each $r \in \mathcal{R}$, we model $x_r(t)$ as a *multiband signal* $x_r(t) \in \mathcal{B}(\mathcal{F}_r)$, where the spectral support $\mathcal{F}_r$ is a finite union of disjoint intervals:

$$\mathcal{F}_r = \bigcup_{n=1}^{N_r} [a_{rn}, b_{rn}), \qquad a_{r1} < b_{r1} < a_{r2} < \cdots < a_{rN_r} < b_{rN_r}. \qquad (3)$$

We model the MIMO channel as a linear and shift-invariant system. Thus, we can write

$$\boldsymbol{y}(t) = \boldsymbol{g} * \boldsymbol{x}(t) = \int_{\mathbb{R}} \boldsymbol{g}(t - \tau) \boldsymbol{x}(\tau) d\tau,$$

where $*$ denotes convolution and

$$\boldsymbol{g}(t) = \begin{pmatrix} g_{0,0}(t) & \cdots & g_{0,R-1}(t) \\ \vdots & \ddots & \vdots \\ g_{P-1,0}(t) & \cdots & g_{P-1,R-1}(t) \end{pmatrix} \in \mathbb{C}^{P \times R}$$

is the impulse response matrix of the channel. Therefore,

$$\boldsymbol{Y}(f) = \boldsymbol{G}(f) \boldsymbol{X}(f), \qquad (4)$$

where $\boldsymbol{X}(f)$, $\boldsymbol{Y}(f)$, and $\boldsymbol{G}(f)$ are the Fourier transforms of $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$, and $\boldsymbol{g}(t)$ respectively. In particular, $\boldsymbol{G}(f)$ is the *channel transfer function matrix*. The channel outputs are sampled at $t = nT$, $n \in \mathbb{Z}$, and we denote these output quantities by $z_p[n] = y_p(nT)$, or in matrix form by

$$\boldsymbol{z}[n] \stackrel{\text{def}}{=} \begin{pmatrix} z_0[n] \\ z_1[n] \\ \vdots \\ z_{P-1}[n] \end{pmatrix} = \boldsymbol{y}(nT), \quad n \in \mathbb{Z}.$$

Then, it is clear that

$$\begin{aligned} \boldsymbol{Z}[\nu] &= \frac{1}{T}\sum_{l\in\mathbb{Z}} \boldsymbol{Y}\Big(\frac{\nu+l}{T}\Big), \quad \nu \in [0,1) \\ &= \frac{1}{T}\sum_{l\in\mathbb{Z}} \boldsymbol{G}\Big(\frac{\nu+l}{T}\Big)\boldsymbol{X}\Big(\frac{\nu+l}{T}\Big), \quad \nu \in [0,1), \end{aligned} \tag{5}$$

where the second line follows from (4). We model the reconstruction block as follows:

$$\tilde{\boldsymbol{x}}(t) = \sum_{n\in\mathbb{Z}} \boldsymbol{h}(t-nT)\boldsymbol{z}[n], \tag{6}$$

where

$$\boldsymbol{h}(t) = \begin{pmatrix} h_{0,0}(t) & \cdots & h_{0,P-1}(t) \\ \vdots & \ddots & \vdots \\ h_{R-1,0}(t) & \cdots & h_{R-1,P-1}(t) \end{pmatrix} \in \mathbb{C}^{R\times P}.$$

It is clear from (6) that the entire MIMO system (consisting of the channel, the samplers and the reconstruction block) is invariant to a time-shift by a multiple of T, *i.e.*

$$\boldsymbol{x}(t) \to \tilde{\boldsymbol{x}}(t) \implies \boldsymbol{x}(t-nT) \to \tilde{\boldsymbol{x}}(t-nT), \quad \forall n \in \mathbb{Z}, t \in \mathbb{R}.$$

Conversely, (6) is the most general linear transformation that allows this invariance. Taking its Fourier transform and rewriting in matrix form yields

$$\tilde{\boldsymbol{X}}(f) = \boldsymbol{H}(f)\boldsymbol{Z}[fT], \quad f \in \mathbb{R}, \tag{7}$$

where $\boldsymbol{H}(f)$, the Fourier transform of $\boldsymbol{h}(t)$, is the *reconstruction filter matrix*. Owing to the periodicity of $\boldsymbol{Z}[\nu]$, we can rewrite (7) as

$$\tilde{\boldsymbol{X}}\Big(f+\frac{l'}{T}\Big) = \boldsymbol{H}\Big(f+\frac{l'}{T}\Big)\boldsymbol{Z}[fT], \quad l' \in \mathbb{Z},\ f \in \Big[0, \frac{1}{T}\Big). \tag{8}$$

We can now rewrite (5) and (8) compactly as

$$\boldsymbol{Z}[fT] = \boldsymbol{\mathcal{G}}(f)\boldsymbol{\mathcal{X}}(f), \tag{9}$$

$$\tilde{\boldsymbol{\mathcal{X}}}(f) = \boldsymbol{\mathcal{H}}(f)\boldsymbol{Z}[fT], \tag{10}$$

for $f \in [0, 1/T)$, where $\boldsymbol{\mathcal{X}}(f)$ and $\tilde{\boldsymbol{\mathcal{X}}}(f)$ are the *modulated input and reconstructed vectors* whose entries are

$$\boldsymbol{\mathcal{X}}_{Rl+r}(f) = X_r\Big(f+\frac{l}{T}\Big), \quad (r,l) \in \mathcal{R}\times\mathbb{Z}, \tag{11}$$

$$\tilde{\boldsymbol{\mathcal{X}}}_{Rl+r}(f), = \tilde{X}_r\Big(f+\frac{l}{T}\Big), \quad (r,l) \in \mathcal{R}\times\mathbb{Z}, \tag{12}$$

while $\boldsymbol{\mathcal{G}}(f)$ and $\boldsymbol{\mathcal{H}}(f)$ for are the *modulated channel and reconstruction matrices* whose entries are

$$\boldsymbol{\mathcal{G}}_{p,Rl+r}(f) = \frac{1}{T}G_{pr}\Big(f+\frac{l}{T}\Big), \quad (p,r,l) \in \mathcal{P}\times\mathcal{R}\times\mathbb{Z}, \tag{13}$$

$$\boldsymbol{\mathcal{H}}_{Rl+r,p}(f) = H_{rp}\Big(f+\frac{l}{T}\Big), \quad (p,r,l) \in \mathcal{P}\times\mathcal{R}\times\mathbb{Z}. \tag{14}$$

Note that, even though these matrices have infinitely many columns or rows, only a finite summation is involved in (9) because the components of $\boldsymbol{X}(f)$ are bandlimited implying that only a finite number of entries in $\boldsymbol{\mathcal{X}}(f)$ are nonzero. In the next section, we seek conditions on the channel and the inputs, that guarantee perfect reconstruction of the input signals, or equivalently, perfect inversion of the channel.

We consider only uniform sampling in this paper. Fortunately, most periodic nonuniform sampling schemes can be expressed as special cases of uniform sampling. To see this, consider the following situation where the p-th channel output $y_p(t)$ is sampled at

$$t \in \Lambda_p = \{nT_p + \lambda_{kp} : k = 0, \ldots, K_p - 1\}.$$

The period of the sampling pattern for the p-th output channel is T_p, and the average sampling density of the p-th output is K_p/T_p. First, consider the case where all the periods are equal, *i.e.*, $T_p = T$. Then, we can write

$$\Lambda_p = \bigcup_{k=0}^{K_p-1} (T\mathbb{Z} + \lambda_{kp}).$$

In other words, Λ_p is composed of a union of K_p uniform sampling sets of density $1/T$. Consider a hypothetical MIMO channel whose transfer function matrix is obtained by performing the following modification to $\boldsymbol{G}(f)$. We replace the p-th row of $\boldsymbol{G}(f)$, namely $\boldsymbol{G}_{p,\bullet}(f)$, by the following K_p rows: $\boldsymbol{G}_{p,\bullet}(f)e^{-j2\pi f\lambda_{kp}}$, $k = 0, \ldots, K_p - 1$. The new channel matrix has $\sum_p K_p$ rows, and the samples of the new outputs taken at $t = nT$ are precisely equal to the samples of the old MIMO channel outputs taken on the periodic nonuniform sampling sets $\{\Lambda_p\}$ and reordered. Next suppose that the different channels have unequal but *commensurate* sampling periods, *i.e.*, that the ratios of sampling periods are rational numbers: $T_p = (m_p/n_p)T$, for some $m_p, n_p \in \mathbb{N}$, and $T \in \mathbb{R}$. In this case, a common period for all the sampling sets $\{\Lambda_p\}$ is $T\prod n_p$, and an argument as before allows us to convert this to uniform sampling of the outputs of a hypothetical MIMO channel. The upshot of this argument is that most periodic nonuniform sampling (except those with non-commensurate periods) may be recast as a uniform sampling problem. Of course the price to pay is that the hypothetical MIMO channel has many more outputs. We illustrate this in the following example.

Example 1: Let $\boldsymbol{G}(f)$ be the channel transfer function matrix of a MIMO channel with $P = 2$ outputs. Let the sampling sets for the channel outputs be as depicted in Figure 3, *i.e.*,

$$\begin{aligned} \Lambda_1 &= \{3n, 3n+0.5 : n \in \mathbb{Z}\}, \\ \Lambda_2 &= \{2n : n \in \mathbb{Z}\}. \end{aligned}$$

These sets are clearly commensurate because sampling periods $T_1 = 3$ and $T_2 = 2$ are such that T_1/T_2 is rational. A common period for the two sampling sets is obviously 6. Indeed, we have

$$\Lambda_1 = \bigcup_{k=0}^{3}(6\mathbb{Z} + \lambda_{k,1}), \quad \{\lambda_{k,1} : k = 0, \ldots, 3\} = \{0, 0.5, 3, 3.5\},$$

$$\Lambda_2 = \bigcup_{k=0}^{2}(6\mathbb{Z} + \lambda_{k,2}), \quad \{\lambda_{k,2} : k = 0, \ldots, 2\} = \{0, 2, 4\}.$$

Hence, the modified channel has six outputs and the rows of its transfer function matrix $\tilde{\boldsymbol{G}}(f)$ are given by

$$\begin{aligned} \tilde{\boldsymbol{G}}_{k,\bullet}(f) &= \boldsymbol{G}_{0,\bullet}(f)e^{-j2\pi f\lambda_{k,1}}, \quad k = 0, 1, 2, 3, \\ \tilde{\boldsymbol{G}}_{k+4,\bullet}(f) &= \boldsymbol{G}_{1,\bullet}(f)e^{-j2\pi f\lambda_{k,2}}, \quad k = 0, 1, 2. \end{aligned}$$

If the outputs of the hypothetical channel are sampled uniformly at $t = 6n$, $n \in \mathbb{Z}$, we essentially obtain a reordered sequence of the samples of the original MIMO channel outputs taken on the samples sets Λ_1 and Λ_2.

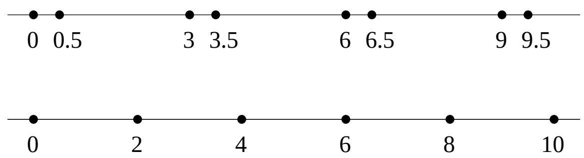

Fig. 3. Commensurate periodic nonuniform sampling sets.

We have shown that commensurate periodic nonuniform sampling is really uniform sampling in disguise, because their equivalence is shown using the above modification trick. Therefore, the study of uniform sampling automatically provides answers to the commensurate periodic nonuniform sampling problem. In the subsequent sections, we present results for uniform MIMO sampling only.

In practice, we would usually only attempt to reconstruct a version of the set of inputs that is uniformly sampled at the corresponding Nyquist rate, and implement $\boldsymbol{H}(f)$ using FIR filters. The continuous-time version could then be reconstructed by a bank of conventional D/A converters on the reconstructed discrete-time signals. In particular, it would be desirable to use a reconstruction filter matrix $\boldsymbol{H}(f)$ that is continuous in f. The reason for this is roughly the following. Recall that a real function on a real compact set can be approximated arbitrarily closely (in the L^∞ sense) by polynomials if the given function is continuous. Similarly, if $\boldsymbol{H}(f)$ is continuous in f, we can approximate the matrix function $\boldsymbol{H}(f)$ arbitrarily closely in the $\mathcal{H}^\infty$ sense (and thus ensure an arbitrarily small worst-case $\mathcal{L}^2$ reconstruction error) by choosing sufficiently long FIR filters. Although we shall not delve into implementation issues in this paper, we do consider both cases (*with* and *without* the continuity requirement imposed on $\boldsymbol{H}(f)$) in the next section, when we derive conditions for perfect reconstruction.

IV. Perfect Reconstruction

A. Preliminaries

We begin with some definitions. Define the following two *spectral index sets* at frequency $f \in [0, 1/T)$:

$$\begin{aligned} \mathcal{K}_f^\circ &= \left\{(r,l) \in \mathcal{R} \times \mathbb{Z} : \left(f + \frac{l}{T}\right) \in \mathcal{F}_r\right\}, \\ \mathcal{K}_f &= \left\{Rl + r : (r,l) \in \mathcal{R} \times \mathbb{Z} \text{ and } \left(f + \frac{l}{T}\right) \in \mathcal{F}_r\right\}. \end{aligned} \tag{15}$$

Recall that the mapping of the pair $(r,l) \in \mathcal{R} \times \mathbb{Z}$ to a single index $Rl + r \in \mathbb{Z}$, is invertible. Hence specifying any one of $\mathcal{K}_f^\circ$ and $\mathcal{K}_f$ uniquely determines the other. Also let $\mathcal{K}_f^c = \mathbb{Z} \backslash \mathcal{K}_f$ denote the complement of $\mathcal{K}_f$. We now have the following proposition.

Proposition 1: Suppose that sets $\mathcal{F}_r$, $r \in \mathcal{R}$ have multiband structure as defined in (3). Then $\mathcal{K}_f$ is piecewise constant on $[0, 1/T)$, *i.e.*, there exists a collection of disjoint intervals $\mathcal{I}_m$ of the form $[\alpha, \beta)$, and sets $\mathcal{K}_m$, $m = 1, \ldots, M$ such that $\mathcal{K}_f = \mathcal{K}_m$, for $f \in \mathcal{I}_m$, and

$$\bigcup_{m=1}^{M} \mathcal{I}_m = \left[0, \frac{1}{T}\right).$$

This result is easily demonstrated by using an argument very similar to the one in [25] for multicoset sampling. Hence, we can write

$$\begin{aligned} &\mathcal{I}_m = [\gamma_m, \gamma_{m+1}), \quad m \in \mathcal{M}, \\ &\gamma_1 < \gamma_2 < \cdots < \gamma_{M+1}, \end{aligned}$$

such that $\gamma_1 = 0$ and $\gamma_{M+1} = 1/T$.

Example 2: Consider a MIMO channel with $R = 2$ inputs, and input spectra $X_0(f)$ and $X_1(f)$ that have supports as illustrated in Figure 2, *i.e.*,

$$\mathcal{F}_0 = [0, 0.4) \cup [0.75, 0.9) \quad \text{and} \quad \mathcal{F}_1 = [0.25, 0.5).$$

Let the sampling period be $T = 4$. For this choice, it is easy to verify that

$$\mathcal{I}_1 = [0, 0.15) \quad \text{and} \quad \mathcal{I}_2 = [0.15, 0.25).$$

Furthermore, (15) and Proposition 1 imply that

$$\mathcal{K}_f^\circ = \begin{cases} \{(0,0), (0,1), (0,3), (1,1)\}, & \text{if } f \in \mathcal{I}_1, \\ \{(0,0), (1,1)\}, & \text{if } f \in \mathcal{I}_2. \end{cases}$$

Therefore $\mathcal{K}_1 = \{0, 2, 6, 3\}$ and $\mathcal{K}_2 = \{0, 3\}$.

In the sequel, we derive the conditions on the channel and the spectral supports $\mathcal{F}_r$ of the channel inputs for the existence of a reconstruction filter matrix $\boldsymbol{H}(f)$ that achieves perfect reconstruction of the inputs. We consider both cases *with* and *without* the continuity requirement imposed on the channel and reconstruction filters. As we shall see later, the continuity of $\boldsymbol{H}(f)$ may also require the continuity of the channel transfer function matrix $\boldsymbol{G}(f)$. Since our analysis in the next subsection relies on the modulated channel and reconstruction matrices $\boldsymbol{\mathcal{G}}(f)$ and $\boldsymbol{\mathcal{H}}(f)$, the following proposition will turn out to be useful.

Proposition 2: If $G_{\bullet,r}(f)$ is continuous on $\overline{\mathcal{F}}_r$ then $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_m}(f)$ is continuous on $\overline{\mathcal{I}}_m$ and the following "boundary condition" holds:

$$\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}}\left(\frac{1}{T}\right) = \boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}\oplus R}(0), \quad \mathcal{K} \subseteq \mathbb{Z}. \tag{16}$$

The quantity $\boldsymbol{H}(f)$ is continuous if and only if the entries of $\boldsymbol{\mathcal{H}}(f)$ are continuous on $[0, 1/T]$ and satisfy the boundary condition

$$\boldsymbol{\mathcal{H}}_{\mathcal{K},\bullet}\left(\frac{1}{T}\right) = \boldsymbol{\mathcal{H}}_{\mathcal{K}\oplus R,\bullet}(0), \quad \mathcal{K} \subseteq \mathbb{Z}. \tag{17}$$

We do not care about $G_{pr}(f)$ outside the closure of the set $\mathcal{F}_r$ because $X_r(f)$ vanishes there. This explains why the conditions for $\boldsymbol{G}(f)$ and $\boldsymbol{H}(f)$ are different in Proposition 2. We omit its proof since it is quite straightforward, following directly from (13) and (14), and the definition of $\mathcal{K}_m$. The boundary conditions imply that the entries of the matrix $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}}(0)$ are shifted versions of those of $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}}(1/T)$, with a similar relationship for $\boldsymbol{\mathcal{H}}_{\mathcal{K},\bullet}$.

Necessary Condition for Perfect Reconstruction

In the next subsection, we use frame theory to derive necessary and sufficient conditions for stable and perfect reconstruction of the channel inputs, but we first present a simple necessary condition. From (15), it is clear that all the nonzero entries of $\boldsymbol{\mathcal{X}}(f)$ are captured in the subvector $\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)$, and hence we can rewrite (9) and (10) as

$$\tilde{\boldsymbol{\mathcal{X}}}_{\mathcal{K}_f}(f) = \boldsymbol{\mathcal{H}}_{\mathcal{K}_f,\bullet}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f), \tag{18}$$

$$\tilde{\boldsymbol{\mathcal{X}}}_{\mathcal{K}_f^c}(f) = \boldsymbol{\mathcal{H}}_{\mathcal{K}_f^c,\bullet}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f). \tag{19}$$

For perfect reconstruction, we require the existence of $\boldsymbol{\mathcal{H}}(f)$ such that $\tilde{\boldsymbol{\mathcal{X}}}(f) = \boldsymbol{\mathcal{X}}(f)$ a.e. It is clear that this would happen if and only if

$$\begin{aligned} \boldsymbol{\mathcal{H}}_{\mathcal{K}_f,\bullet}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f) &= \boldsymbol{I}_{|\mathcal{K}_f|} \quad \text{a.e.}, \\ \boldsymbol{\mathcal{H}}_{\mathcal{K}_f^c,\bullet}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f) &= \boldsymbol{0} \quad \text{a.e.}, \end{aligned} \tag{20}$$

which can be expressed more compactly as

$$\boldsymbol{\mathcal{H}}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f) = \boldsymbol{I}_{\bullet,|\mathcal{K}_f|}, \tag{21}$$

Since $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f) \in \mathbb{C}^{P\times|\mathcal{K}_f|}$, we require that $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)$ have full column rank a.e. This condition guarantees that a solution (possibly nonunique) to (21) exists. In view of Proposition 1, we now obtain the following necessary condition for perfect reconstruction:

$$\operatorname{rank}\left(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_m}(f)\right) = |\mathcal{K}_m|, \quad \text{a.e. } f \in \mathcal{I}_m. \tag{22}$$

However, this condition does not address the issue of stability of reconstruction, and hence may be insufficient.

Example 3: The necessary conditions reduce to a familiar form for the special case of a single-input, single-output (SISO) channel, with $R = P = 1$. This case then corresponds to (single channel) deconvolution of a multiband signal $x \in \mathcal{B}(\mathcal{F})$ from the sampled output y. The Fourier transforms $X(f)$ and $Y(f)$ of the channel input and the output, respectively, and the channel transfer function $G(f)$ are all scalar in this case . It follows that the spectral index set defined in (15) reduces to $\mathcal{K}_f = \left\{ l : f + \frac{l}{T} \in \mathcal{F} \right\}$, and the modulated channel matrix $\boldsymbol{\mathcal{G}}(f)$ has only one row. Hence, the necessary condition for perfect reconstruction in (22) is equivalent to the following set of conditions

$$|\mathcal{K}_f| \leq 1 \quad \text{and} \quad G(f) \neq 0, \quad f \in \mathcal{F}. \tag{23}$$

The first condition says that there must be no aliasing of $\mathcal{F}$ due to sampling, and the second one says that the channel transfer function cannot have any nulls on the set $\mathcal{F}$.

These conditions can be easily re-derived "from first principles" as follows. Suppose that any $x \in \mathcal{B}(\mathcal{F})$ can be reconstructed from the samples of y. Then x can also be reconstructed from y itself, but this is only possible if the relationship (4), $Y(f) = G(f)X(f)$, can be inverted. The necessary condition on $G(f)$ then follows. Returning to the assumption that x can be reconstructed from the samples of y it follows that y can also be obtained using (4). However, as we know from classical results on uniform multiband sampling, y can only be obtained from its uniform samples if its spectrum is not aliased. Noting that, owing to the condition on $G(f)$, $Y(f)$ is supported on $\mathcal{F}$, the non-aliasing condition in (23) follows.

These conditions do not, however, guarantee stability of inversion. For instance, if $G(f)$ takes arbitrarily small or large values for $f \in \mathcal{F}$, we cannot invert (4) in a stable way. We study the stability and present necessary and sufficient conditions immediately following this example.

Finally, specializing the example further to pure single channel sampling, consider the case $G(f) = 1$. In this case, the necessary condition on $G(f)$ holds trivially. The only other necessary condition is the no-aliasing condition $|\mathcal{K}_f| \leq 1$ for perfect inversion. Incidentally, this condition is both necessary and sufficient for stable inversion, as we know for the classical problem of uniform multiband sampling.

B. Stable sampling

The MIMO channel can be viewed as a linear transformation from the class of input signals to the space of its samples. The condition in (22) on the channel and the input signals is necessary for stable perfect reconstruction. However, it does not suffice because it does not answer the important question regarding stability of the reconstruction. In this section, we shall use frame theory to study the stability of the MIMO sampling problem. Recall the definition of a frame:

Definition 1: Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\{\psi_n\} \subseteq \mathcal{H}$ is a frame if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_n |\langle \psi_n, x \rangle|^2 \leq B\|x\|^2,$$

for all $x \in \mathcal{H}$. If $A = B$, then the frame is a tight frame.
The frame operator S, defined as

$$Sx = \sum_n \langle x, \psi_n \rangle \psi_n, \quad \forall x \in \mathcal{H},$$

is a bounded linear operator satisfying $AI \leq S \leq BI$, where I is the identity operator. Define $\tilde{\psi}_n = S^{-1}\psi_n$. Then $\{\tilde{\psi}_n\}$ is also a frame (the *dual frame*) for $\mathcal{H}$ with frame bounds B^{-1} and A^{-1}, and any $x \in \mathcal{H}$ can be expressed as

$$x = \sum_n \langle x, \tilde{\psi}_n \rangle \psi_n = \sum_n \langle x, \psi_n \rangle \tilde{\psi}_n. \tag{24}$$

In the context of MIMO sampling, the relevant Hilbert space is the class of input signals:

$$\mathcal{H} = \mathcal{B}(\mathcal{F}_1) \times \cdots \times \mathcal{B}(\mathcal{F}_R).$$

The inner product and norm on $\mathcal{H}$ are defined as

$$\langle \boldsymbol{x}, \boldsymbol{w} \rangle = \int_{\mathbb{R}} \boldsymbol{w}^H(t)\boldsymbol{x}(t)dt, \quad \boldsymbol{x}, \boldsymbol{w} \in \mathcal{H},$$
$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}.$$

We now present an important definition for the stability of MIMO sampling (see [29]):

Definition 2: The MIMO sampling scheme is called *stable* if there exist constants $A, B > 0$ such that

$$A\|\boldsymbol{x}\|^2 \leq \sum_{n \in \mathbb{Z}} \|\boldsymbol{z}[n]\|^2 \leq B\|\boldsymbol{x}\|^2, \tag{25}$$

for all $\boldsymbol{x}(t) \in \mathcal{H}$.
To see that (25) allows stable reconstruction, we recast it in a frame theoretic form. Define the diagonal matrix

$$\boldsymbol{J}(f) = \operatorname{diag}\left(\chi(f \in \mathcal{F}_1), \ldots, \chi(f \in \mathcal{F}_R)\right). \tag{26}$$

Then we have

$$\boldsymbol{J}(f)\boldsymbol{X}(f) = \boldsymbol{X}(f) \tag{27}$$

because $X_r(f)$ is supported on $\mathcal{F}_r$. In view of (27), we can rewrite $z_p[n]$ as

$$\begin{aligned} z_p[n] = y_p(nT) &= \int_{\mathbb{R}} e^{j2\pi f nT} \boldsymbol{G}_{p,\bullet}(f)\boldsymbol{X}(f)df \\ &= \int_{\mathbb{R}} e^{j2\pi f nT} \boldsymbol{G}_{p,\bullet}(f)\boldsymbol{J}(f)\boldsymbol{X}(f)df \\ &= \int_{\mathbb{R}} \boldsymbol{\Psi}^H_{pn}(f)\boldsymbol{X}(f)df, \end{aligned} \tag{28}$$

where

$$\begin{aligned} &\boldsymbol{\Psi}^H_{pn}(f) = e^{j2\pi f nT} \boldsymbol{G}_{p,\bullet}(f)\boldsymbol{J}(f) \\ \Longleftrightarrow\ &\boldsymbol{\psi}_{pn}(t) = \int_{\mathbb{R}} \boldsymbol{J}(f)\boldsymbol{G}^H_{p,\bullet}(f)e^{j2\pi f(t-nT)}df \end{aligned} \tag{29}$$

for $(p, n) \in \mathcal{P} \times \mathbb{Z}$. It is clear that $\boldsymbol{\psi}_{pn} \in \mathcal{H}$. Using Parseval's theorem and (29) we conclude that

$$\langle \boldsymbol{x}, \boldsymbol{\psi}_{pn} \rangle = \int_{\mathbb{R}} \boldsymbol{\psi}^H_{pn}(t)\boldsymbol{x}(t)dt = \int_{\mathbb{R}} \boldsymbol{\Psi}^H_{pn}(f)\boldsymbol{X}(f)df = z_p[n].$$

Thus $z_p[n]$ can be expressed as an inner product of $\boldsymbol{x}$ and $\boldsymbol{\psi}_{pn} \in \mathcal{H}$, and consequently, (25) is equivalent to the condition that $\{\boldsymbol{\psi}_{pn}\}$ forms a frame for $\mathcal{H}$. Suppose we denote its dual frame by $\{\tilde{\boldsymbol{\psi}}_{np} : n \in \mathbb{Z},\ p \in \mathcal{P}\}$, then (24) yields the following reconstruction formula:

$$\boldsymbol{x} = \sum_{n \in \mathbb{Z}} \sum_{p \in \mathcal{P}} \langle \boldsymbol{x}, \boldsymbol{\psi}_{np} \rangle \tilde{\boldsymbol{\psi}}_{np} = \sum_{n \in \mathbb{Z}} \sum_{p \in \mathcal{P}} z_p[n] \tilde{\psi}_{np}.$$

As shown in [29] the implication of stability of reconstruction is that errors in the inputs or the sampled outputs cannot produce arbitrarily large errors in the reconstructed inputs. The ratio $K = \sqrt{B/A} \geq 1$ is called the *condition number* of the sampling scheme, and K^2 is a bound on the amplification of the normalized error energy due to the reconstruction filters.

C. Conditions for perfect reconstruction

Let ess inf and ess sup denote the essential infimum and supremum, *i.e.*,

$$\text{ess}\inf g(t) = \sup\{\gamma : g(t) \geq \gamma \text{ a.e.}\},$$
$$\text{ess}\sup g(t) = \inf\{\gamma : g(t) \leq \gamma \text{ a.e.}\}.$$

for any real function g. Our next result provides necessary and sufficient conditions on the channel matrix for stable MIMO reconstruction.

Theorem 1: The frame bounds for the MIMO sampling problem are given by

$$A = T \operatorname*{ess\,inf}_{f\in[0,1/T]} \lambda_{\min}\big(\boldsymbol{\mathcal{G}}^H_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\big), \tag{30}$$

$$B = T \operatorname*{ess\,sup}_{f\in[0,1/T]} \lambda_{\max}\big(\boldsymbol{\mathcal{G}}^H_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\big). \tag{31}$$

In particular, $A > 0$ and $B < \infty$ are necessary and sufficient conditions for stable reconstruction of the MIMO inputs.

Proof: We need to compute

$$A = \inf_{\boldsymbol{x}\in\mathcal{B}} \sum_{n\in\mathbb{Z}} \|\boldsymbol{z}[n]\|^2 \quad \text{and} \quad B = \inf_{\boldsymbol{x}\in\mathcal{B}} \sum_{n\in\mathbb{Z}} \|\boldsymbol{z}[n]\|^2, \tag{32}$$

where $\mathcal{B}$ is the set of input signals of unit combined energy:

$$\mathcal{B} = \Big\{\boldsymbol{x} \in \mathcal{H} : \|\boldsymbol{x}\| = 1\Big\}. \tag{33}$$

First observe that

$$\begin{aligned}\|\boldsymbol{x}\|^2 &= \int_{\mathbb{R}} \|\boldsymbol{x}(t)\|^2 dt = \int_{\mathbb{R}} \|\boldsymbol{X}(f)\|^2 df \\ &\overset{(a)}{=} \int_{[0,\frac{1}{T}]} \|\boldsymbol{\mathcal{X}}(f)\|^2 df \\ &\overset{(b)}{=} \int_{[0,\frac{1}{T}]} \|\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)\|^2 df,\end{aligned} \tag{34}$$

where the norms on the right hand side of (34) are the Euclidean norms. The equality (a) above follows from (11), and (b) follows because $\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)$ captures all the nonzero entries of $\boldsymbol{\mathcal{X}}(f)$. Next

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \|\boldsymbol{z}[n]\|^2 &= \int_{\nu\in[0,1]} \|\boldsymbol{z}[\nu]\|^2 d\nu \\ &\overset{(a)}{=} T\int_{[0,\frac{1}{T}]} \|\boldsymbol{z}[TV]\|^2 df \\ &\overset{(b)}{=} T\int_{[0,\frac{1}{T}]} \|\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)\|^2 df,\end{aligned} \tag{35}$$

where (a) is obtained by a change of variables, and (b) from (9) and the fact that $\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)$ captures all the nonzero entries of $\boldsymbol{\mathcal{X}}(f)$. Therefore (32), (33), (34), and (35) yield

$$\begin{aligned}A &= \inf T\int_{[0,\frac{1}{T}]} \|\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)\|^2 df \\ &\text{s.t.} \quad \int_{[0,\frac{1}{T}]} \|\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)\|^2 df = 1, \\ B &= \sup T\int_{[0,\frac{1}{T}]} \|\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_f}(f)\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)\|^2 df \\ &\text{s.t.} \quad \int_{[0,\frac{1}{T}]} \|\boldsymbol{\mathcal{X}}_{\mathcal{K}_f}(f)\|^2 df = 1.\end{aligned}$$

Now the claimed results in (30) and (31) follow immediately. ■

Note that a simple necessary condition for perfect reconstruction is that $P \geq |\mathcal{K}_m|$ for each $m \in \mathcal{M}$. Clearly, multiple solutions $\boldsymbol{\mathcal{H}}(f)$ exist to (21) if $P > |\mathcal{K}_m|$ for some m. The average sampling density for this sampling scheme is P/T. Now, (15) implies that

$$|\mathcal{K}_f| = \sum_{r\in\mathcal{R}} \sum_{l\in\mathbb{Z}} \chi(f + l/T \in \mathcal{F}_r).$$

Hence

$$\int_{[0,1/T]} |\mathcal{K}_f| df = \sum_{r=1}^{R} \int_{R} \chi(f \in \mathcal{F}_r) = \sum_{r=1}^{R} \mu(\mathcal{F}_r), \tag{36}$$

where $\mu(\cdot)$ denotes the Lebesgue measure. Suppose that $P = |\mathcal{K}_m|$ for all m, then (36) reduces to

$$\frac{P}{T} = \sum_{r=1}^{R} \mu(\mathcal{F}_r).$$

This value for the total sampling density coincides with the minimum density required for stable MIMO sampling [29] using any sampling scheme for the channel outputs, whether uniform or not. Also note that we have uniqueness of the reconstruction filters when $P = |\mathcal{K}_m|$.

The following corollary to Theorem 1 provides a simpler sufficient condition for the stability of the MIMO sampling scheme.

Corollary 1: Suppose that $\boldsymbol{G}(f)$ is such that $G_{pr}(f)$ is continuous for $f \in \overline{\mathcal{F}}_r$, and $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_m}(f)$ has full column rank for all $m \in \mathcal{M}$, $f \in \overline{\mathcal{I}}_m = [\gamma_m, \gamma_{m+1}]$. Then the MIMO sampling scheme is stable, *i.e.*, $\{\boldsymbol{\psi}_{pn}\}$ forms a frame.

Proof: By Proposition 2, we have continuity of $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_m}(f)$ on the compact set $\overline{\mathcal{I}}_m$. Therefore, both the smallest and the largest eigenvalues of $\boldsymbol{\mathcal{G}}^H_{\bullet,\mathcal{K}_m}(f)\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_m}(f)$ are continuous functions on $\overline{\mathcal{I}}_m$. Since the smallest eigenvalue is strictly positive for all $f \in [0, 1/T]$ by hypothesis, it follows that the infimum in (30) is attained, implying that $A > 0$. Similarly, $B < \infty$ because the supremum in (31) is attained. ■

We illustrate the MIMO sampling result of Theorem 1 for a simple MIMO channel.

Example 4: Consider a MIMO channel with $R = 2$ inputs and $P = 4$ outputs having the following transfer function matrix:

$$\boldsymbol{G}(f) = \begin{pmatrix} 1 & 1 \\ 1 & 1 + e^{-j2\pi f} \\ e^{-j2\pi f} & 0.25 + e^{-j4\pi f} \\ 1 + 0.5e^{-j2\pi f} & 1 + e^{-j4\pi f} \end{pmatrix}.$$

Let the input spectra $X_0(f)$ and $X_1(f)$ have supports as illustrated in Figure 2, *i.e.*,

$$\mathcal{F}_0 = [0, 0.4) \cup [0.75, 0.9) \quad \text{and} \quad \mathcal{F}_1 = [0.25, 0.5).$$

Each output is a multiband signal supported on $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 = [0, 0.5) \cup [0.75, 0.9)$. Hence the Nyquist rate for sampling each output is $\mu(\mathcal{F}) = 0.65$. However, we demonstrate in this example that we do not need to sample each output at the Nyquist rate to achieve perfect reconstruction. Let the sampling period be $T = 4$. For this choice, we have seen in Example 2 that $\mathcal{I}_1 = [0, 0.15)$ and $\mathcal{I}_2 = [0.15, 0.25)$. and $\mathcal{K}_1 = \{0, 2, 6, 3\}$, $\mathcal{K}_2 = \{0, 3\}$. A simple calculation yields expressions for $\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_m}(f)$ shown in (37).

It can be verified numerically that

$$\operatorname{rank}\big(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_1}(f)\big) = 4, \quad \forall f \in \overline{\mathcal{I}_1},$$
$$\operatorname{rank}\big(\boldsymbol{\mathcal{G}}_{\bullet,\mathcal{K}_2}(f)\big) = 2, \quad \forall f \in \overline{\mathcal{I}_2}.$$

Since $\boldsymbol{G}(f)$ is continuous, we conclude using Corollary 1, that stable perfect reconstruction of the inputs is possible from the channel output samples. Hence, a sampling rate of 0.25, instead of the Nyquist rate

$$\mathcal{G}_{\bullet,\mathcal{K}_1}(f) = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1+e^{-j2\pi(f+1/4)} \\ e^{-j2\pi f} & e^{-j2\pi(f+1/4)} & e^{-j2\pi(f+3/4)} & 0.25+e^{-j4\pi(f+1/4)} \\ 1+0.5e^{-j2\pi f} & 1+0.5e^{-j2\pi(f+1/4)} & 1+0.5e^{-j2\pi(f+3/4)} & 1+e^{-j4\pi(f+1/4)} \end{pmatrix},$$
$$\mathcal{G}_{\bullet,\mathcal{K}_2}(f) = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1+e^{-j2\pi(f+1/4)} \\ e^{-j2\pi f} & 0.25+e^{-j4\pi(f+1/4)} \\ 1+0.5e^{-j2\pi f} & 1+e^{-j4\pi(f+1/4)} \end{pmatrix}. \quad (37)$$

of 0.65 suffices for perfect stable reconstruction of the channel inputs. However, as we shall see in Example 6, the reconstruction filter matrix $\boldsymbol{H}(f)$ would necessarily have to be discontinuous. Finally, note that the total combined sampling density of the outputs is $P/T = 1$, while the minimum density, as dictated by [29] is $\mu(\mathcal{F}_1) + \mu(\mathcal{F}_2) = 0.8$.
In Example 4, we showed that the combined sampling density of 1 is achievable, but the lower bound on this density is 0.8. Therefore, we could potentially find a nonuniform MIMO sampling scheme that closes the gap. In fact, this is precisely what we are going to show in the following example.

Example 5: Let the inputs signal characteristics and the channel transfer function matrix be the same as in Example 4. We show that, using a proper nonuniform sampling strategy at the outputs, we can achieve the minimum combined sampling rate for all the output channels equal to the sum of measures of the input spectral supports [29]. Let the channel outputs be sampled on the sets $\Lambda_p = \{20n + \lambda_{kp} : k = 0, \ldots, K_p - 1\}$, where $(K_1, K_2, K_3, K_4) = (0, 3, 5, 8)$ and

$$\{\lambda_{kp} : k = 0, \ldots, K_p - 1\} = \begin{cases} \emptyset & \text{if } p = 0, \\ \{1, 8, 14\} & \text{if } p = 1, \\ \{2, 5, 8, 13, 18\} & \text{if } p = 2, \\ \{0, 2, 4, 5, 7, 8, 14, 17\} & \text{if } p = 3. \end{cases}$$

Evidently, these are all periodic nonuniform sampling sets having a common period of $T = 20$, and consisting of 16 cosets in all. Hence, the modified MIMO channel has a transfer function matrix $\tilde{\boldsymbol{G}}(f)$ of size 16×2, and its rows can be worked out as in Example 1. Since the band edges of $\mathcal{F}_1$ and $\mathcal{F}_2$ are all multiples of 0.05, we trivially obtain $M = 1, \mathcal{I}_1 = [0, 0.05)$, and

$$\mathcal{K}_1 = \{0, 2, 4, 6, 8, 10, 12, 14, 30, 32, 34\} \cup \{11, 13, 15, 17, 19\}.$$

Now, $\tilde{\mathcal{G}}_{\bullet,\mathcal{K}_1}(f)$ is a continuous 16×16 matrix, whose rank is verifiable to be 16 for all f. By Corollary 1, we conclude that stable and perfect reconstruction of the channel inputs is possible from these periodic nonuniform MIMO samples. In fact, the stability bounds are $A = 8.0724 \times 10^{-4}$ and $B = 3.6833$, implying that the condition number $K = \sqrt{B/A} = 67.5487$. The sampling density of Λ_p is $d_p = K_p/T$, so that

$$(d_1, d_2, d_3, d_4) = (0, 0.15, 0.25, 0.5)$$

is an achievable point in density region for stable sampling. Obviously, the densities (d_1, d_2, d_3, d_4) must meet all the necessary conditions for stable sampling derived in [29]. In particular, the total combined sampling rate of all the outputs is $16/T = 0.8$, which is precisely equal to the minimum joint sampling density required, namely $\mu(\mathcal{F}_1) + \mu(\mathcal{F}_2)$. This example also illustrates, in the MIMO sampling context, that the combined sampling rate (of 0.8) can be smaller than even the Nyquist rate for one of the inputs (X_0) alone (0.9 in this example). This situation is similar to that observed in sub-Nyquist nonuniform sampling of single channel multiband signals whose spectral support does not tile under translation [25]. Finally, we learn from this example that we need not sample the different outputs at the same rate. In fact, one of the channels is not sampled at all, unlike in Example 4 where, due to uniform sampling, we required samples from all channel outputs.

D. Existence of continuous solutions

Theorem 1 does not guarantee the existence of a continuous filter matrix $\boldsymbol{H}(f)$ which, as we have seen earlier, may be desirable from an implementation point of view. The following theorem shows that, under a stronger set of conditions, we can guarantee the existence of a continuous filter matrix $\boldsymbol{H}(f)$. We begin with a lemma.

Lemma 1: Let $\boldsymbol{C}(f) \in \mathbb{C}^{p \times q}$ (with $p \geq q$) and $\boldsymbol{D}(f) \in \mathbb{C}^{r \times q}$ be matrix-valued continuous functions of $f \in [\alpha, \beta]$ such that $\operatorname{rank} \boldsymbol{C}(f) = q$ for all $f \in [\alpha, \beta]$, and let $\boldsymbol{E}_\alpha, \boldsymbol{E}_\beta \in \mathbb{C}^{r \times p}$ be matrices satisfying

$$\boldsymbol{E}_\alpha \boldsymbol{C}(\alpha) = \boldsymbol{D}(\alpha) \quad \text{and} \quad \boldsymbol{E}_\beta \boldsymbol{C}(\beta) = \boldsymbol{D}(\beta).$$

Then there exists a continuous $\boldsymbol{E}(f) \in \mathbb{C}^{r \times p}$ such that $\boldsymbol{E}(f)\boldsymbol{C}(f) = \boldsymbol{D}(f)$ for all $f \in [\alpha, \beta]$, and that $\boldsymbol{E}(\alpha) = \boldsymbol{E}_\alpha$ and $\boldsymbol{E}(\beta) = \boldsymbol{E}_\beta$ are satisfied.
The proof of Lemma 1 can be found in the Appendix. We can now derive the conditions for the existence of continuous reconstruction filters that achieve perfect reconstruction.

Theorem 2: Suppose that the MIMO transfer function matrix $\boldsymbol{G}(f)$ is such that $G_{pr}(f)$ is continuous for $f \in \overline{\mathcal{F}}_r$. Then there exists a reconstruction filter matrix $\boldsymbol{H}(f)$ continuous in f, that achieves stable and perfect reconstruction of the MIMO channel inputs if and only if

$$\operatorname{rank}\left(\mathcal{G}_{\bullet,\mathcal{K}_m}(f)\right) = |\mathcal{K}_m|, \quad \forall f \in \operatorname{int} \mathcal{I}_m = (\gamma_m, \gamma_{m+1}), \quad (38)$$
$$\operatorname{rank}\left(\mathcal{G}_{\bullet,\mathcal{J}_m}(\gamma_m)\right) = |\mathcal{J}_m|, \quad m \in \mathcal{M}. \quad (39)$$

where

$$\mathcal{J}_m = \mathcal{K}_m \cup \mathcal{K}_{m-1}, \quad m = 2, \ldots, M,$$
$$\mathcal{J}_1 = \mathcal{K}_1 \cup (\mathcal{K}_M \oplus R). \quad (40)$$

Proof: First note that the hypotheses in this theorem are stronger than those of Corollary 1. Thus stable reconstruction is guaranteed. We shall first prove the necessity of (38) and (39). The first condition in (20) states that

$$\mathcal{H}_{\mathcal{K}_f,\bullet}(f)\mathcal{G}_{\bullet,\mathcal{K}_f}(f) = \boldsymbol{I}_{|\mathcal{K}_f|} \quad \text{a.e.} \quad (41)$$

So, suppose that $\boldsymbol{H}(f)$ is a continuous solution of (41), then Proposition 2 implies that $\mathcal{H}_{\mathcal{K}_m,\bullet}(f)$ and $\mathcal{G}_{\bullet,\mathcal{K}_m}(f)$ are continuous functions in the interior of $\mathcal{I}_m$, and, in fact, (41) must hold for all $f \in \operatorname{int} \mathcal{I}_m$, not just a.e., because both sides of (41) are continuous functions. Now (38) follows immediately. Next, letting $f \downarrow \gamma_1 = 0$ in (20) and using the continuity of $\mathcal{H}(f)$ gives us

$$\mathcal{H}_{\mathcal{K}_1,\bullet}(0)\mathcal{G}_{\bullet,\mathcal{K}_1}(0) = \boldsymbol{I}_{|\mathcal{K}_1|}, \quad \mathcal{H}_{\mathcal{K}_1^c,\bullet}(0)\mathcal{G}_{\bullet,\mathcal{K}_1}(0) = \boldsymbol{0}, \quad (42)$$

while letting $f \uparrow \gamma_{M+1} = 1/T$ in (20) instead, and using (16) and (17), we obtain

$$\begin{aligned} \mathcal{H}_{\mathcal{K}_M \oplus R,\bullet}(0)\mathcal{G}_{\bullet,\mathcal{K}_M \oplus R}(0) &= \boldsymbol{I}_{|\mathcal{K}_M|}, \\ \mathcal{H}_{\mathcal{K}_M^c \oplus R,\bullet}(0)\mathcal{G}_{\bullet,\mathcal{K}_M \oplus R}(0) &= \boldsymbol{0}. \end{aligned} \tag{43}$$

Combining (42) and (43), we obtain the following set of necessary conditions:

$$\mathcal{H}_{\mathcal{J}_1,\bullet}(0)\mathcal{G}_{\bullet,\mathcal{J}_1}(0) = \boldsymbol{I}_{|\mathcal{J}_M|}, \quad \mathcal{H}_{\mathcal{J}_1^c,\bullet}(0)\mathcal{G}_{\bullet,\mathcal{J}_1}(0) = \boldsymbol{0}. \tag{44}$$

where $\mathcal{J}_1 = \mathcal{K}_1 \cup (\mathcal{K}_M \oplus R)$. Using a similar continuity argument in the vicinity of γ_m, for $m = 2, \ldots, M$ yields:

$$\mathcal{H}_{\mathcal{J}_m,\bullet}(\gamma_m)\mathcal{G}_{\bullet,\mathcal{J}_m}(\gamma_m) = \boldsymbol{I}_{|\mathcal{J}_m|}, \quad \mathcal{H}_{\mathcal{J}_m^c,\bullet}(\gamma_m)\mathcal{G}_{\bullet,\mathcal{J}_m}(\gamma_m) = \boldsymbol{0}. \tag{45}$$

where $\mathcal{J}_m = \mathcal{K}_m \cup \mathcal{K}_{m-1}$. Hence (39) is necessary to meet conditions in (44) and (45).

To prove sufficiency of (38) and (39), we construct an appropriate reconstruction matrix $\mathcal{H}(f)$ that is continuous in f and satisfies the boundary condition in (17), as well as the defining reconstruction conditions in (20). We first define the function $\mathcal{H}(f)$ on the following finite set of frequencies $\{\gamma_m : m \in \mathcal{M}\}$:

$$\mathcal{H}_{\mathcal{J}_m,\bullet}(\gamma_m) = \mathcal{G}^{\dagger}_{\bullet,\mathcal{J}_m}(\gamma_m), \quad \mathcal{H}_{\mathcal{J}_m^c,\bullet}(\gamma_m) = \boldsymbol{0}. \tag{46}$$

Then to satisfy (17), we define

$$\begin{aligned} \mathcal{H}_{\mathcal{J}_{M+1},\bullet}\left(\frac{1}{T}\right) &= \mathcal{H}_{\mathcal{J}_1,\bullet}(0) = \mathcal{G}^{\dagger}_{\bullet,\mathcal{J}_1}(0), \\ \mathcal{H}_{\mathcal{J}^c_{M+1},\bullet}\left(\frac{1}{T}\right) &= \mathcal{H}_{\mathcal{J}^c_{M+1}\oplus R,\bullet}(0) = \mathcal{H}_{\mathcal{J}_1^c,\bullet}(0) = \boldsymbol{0} \end{aligned}$$

where

$$\mathcal{J}_{M+1} \stackrel{\text{def}}{=} \mathcal{J}_1 \ominus R = (\mathcal{K}_1 \ominus R) \cup \mathcal{K}_M.$$

Therefore, using (16), we now have

$$\mathcal{H}_{\mathcal{J}_{M+1},\bullet}\left(\frac{1}{T}\right) = \mathcal{G}^{\dagger}_{\bullet,\mathcal{J}_{M+1}}\left(\frac{1}{T}\right), \quad \mathcal{H}_{\mathcal{J}^c_{M+1},\bullet}\left(\frac{1}{T}\right) = \boldsymbol{0}. \tag{47}$$

To complete the proof, it suffices to construct a *continuous extension* $\mathcal{H}(f)$ on $[0, 1/T]$ that satisfies (20), (46), and (47). With the intention of applying Lemma 1, define the following quantities:

$$\alpha = \gamma_m, \quad \boldsymbol{E}_\alpha = \begin{pmatrix} \mathcal{H}_{\mathcal{K}_m,\bullet}(\alpha) \\ \mathcal{H}_{(\mathcal{J}_m \cup \mathcal{J}_{m+1})\setminus\mathcal{K}_m,\bullet}(\alpha) \end{pmatrix}, \quad \boldsymbol{C}(f) = \mathcal{G}_{\bullet,\mathcal{K}_m}(f),$$

$$\beta = \gamma_{m+1}, \quad \boldsymbol{E}_\beta = \begin{pmatrix} \mathcal{H}_{\mathcal{K}_m,\bullet}(\beta) \\ \mathcal{H}_{(\mathcal{J}_m \cup \mathcal{J}_{m+1})\setminus\mathcal{K}_m,\bullet}(\beta) \end{pmatrix}, \quad \boldsymbol{D}(f) = \begin{pmatrix} \boldsymbol{I}_{|\mathcal{K}_m|} \\ \boldsymbol{0} \end{pmatrix}.$$

Observe that $\boldsymbol{C}(f) = \mathcal{G}_{\bullet,\mathcal{K}_m}(f)$ has full column rank for $f \in [\gamma_m, \gamma_{m+1}]$. Moreover, using $\mathcal{K}_m \subseteq \mathcal{J}_m$, and $\mathcal{K}_m \subseteq \mathcal{J}_{m+1}$, it follows from (46) that

$$\begin{aligned} \boldsymbol{E}_\alpha \boldsymbol{C}(\alpha) &= \begin{pmatrix} \boldsymbol{I}_{|\mathcal{K}_m|} \\ \boldsymbol{0} \end{pmatrix} = \boldsymbol{D}(\alpha), \\ \boldsymbol{E}_\beta \boldsymbol{C}(\beta) &= \begin{pmatrix} \boldsymbol{I}_{|\mathcal{K}_m|} \\ \boldsymbol{0} \end{pmatrix} = \boldsymbol{D}(\beta). \end{aligned}$$

Thus, we have verified all the technical conditions required in Lemma 1, and we are guaranteed a continuous solution

$$\boldsymbol{E}(f) = \begin{pmatrix} \mathcal{H}_{\mathcal{K}_m,\bullet}(f) \\ \mathcal{H}_{(\mathcal{J}_m \cup \mathcal{J}_{m+1})\setminus\mathcal{K}_m,\bullet}(f) \end{pmatrix}$$

that meets the desired boundary conditions and satisfies

$$\begin{aligned} \mathcal{H}_{\mathcal{K}_m,\bullet}(f)\mathcal{G}_{\bullet,\mathcal{K}_m}(f) &= \boldsymbol{I}_{|\mathcal{K}_f|}, \\ \mathcal{H}_{(\mathcal{J}_m \cup \mathcal{J}_{m+1})\setminus\mathcal{K}_m,\bullet}(f)\mathcal{G}_{\bullet,\mathcal{K}_m}(f) &= \boldsymbol{0} \end{aligned} \tag{48}$$

for $f \in [\gamma_m, \gamma_{m+1}]$. We also define

$$\mathcal{H}_{(\mathcal{J}_m \cup \mathcal{J}_{m+1})^c,\bullet}(f) = \boldsymbol{0}, \quad f \in [\gamma_m, \gamma_{m+1}]. \tag{49}$$

Therefore, (48) and (49) provide us with a continuous extension for $\mathcal{H}(f)$ on $[\gamma_m, \gamma_{m+1}]$ that satisfies (20) for each $m \in \mathcal{M}$, and hence for the entire interval $[0, 1/T]$. Because $\mathcal{H}(f)$ by construction also satisfies the boundary conditions (17), the continuity of $\boldsymbol{H}(f)$ follows by Proposition 2. ■

Remarks:

1. A simple necessary condition for perfect reconstruction using continuous reconstruction filters is that $P \geq \max_m |\mathcal{J}_m|$.
2. Although the continuity of the entries of $\boldsymbol{G}(f)$ was essential in the above proof, it is not strictly necessary as it is possible to carefully construct examples where a continuous $\boldsymbol{H}(f)$ exists even though $\boldsymbol{G}(f)$ may be discontinuous.

The following example illustrates Theorem 2

Example 6: Assume that $R = 2$, $T = 4$, and that the input spectra have the same form as in Examples 2 and 4. Then $\mathcal{K}_1 = \{0, 2, 6, 3\}$ and $\mathcal{K}_2 = \{0, 3\}$. In addition the index sets defined in (40) are

$$\begin{aligned} \mathcal{J}_1 &= \mathcal{K}_1 \cup (\mathcal{K}_2 \oplus 2) = \{0, 2, 6, 3, 5\}, \\ \mathcal{J}_2 &= \mathcal{K}_2 \cup \mathcal{K}_1 = \{0, 2, 3, 6\}. \end{aligned}$$

Hence, $P \geq \max_m |\mathcal{J}_m| = 5$ is necessary for the existence of a continuous $\boldsymbol{H}(f)$, and clearly, the transfer function matrix $\boldsymbol{G}(f)$ of Example 4 does not suffice. So let us append a new row beneath the last row of $\boldsymbol{G}(f)$, thereby making the MIMO channel a two-input five-output channel:

$$\boldsymbol{G}(f) = \begin{pmatrix} 1 & 1 \\ 1 & 1 + e^{-j2\pi f} \\ e^{-j2\pi f} & 0.25 + e^{-j4\pi f} \\ 1 + 0.5e^{-j2\pi f} & 1 + e^{-j4\pi f} \\ 0.25 + e^{-j4\pi f} & e^{-j2\pi f} \end{pmatrix}.$$

The rank condition in (38) holds because the matrix $\mathcal{G}_{\bullet,\mathcal{K}_m}(f)$ of Example 4 has full column rank, and adding an extra row to $\boldsymbol{G}(f)$ (and hence to $\mathcal{G}(f)$ also) does not lower the column rank of $\mathcal{G}_{\bullet,\mathcal{K}_m}(f)$. Figure 4 depicts the smallest and largest eigenvalues of the matrix

$$\boldsymbol{S}(f) = \mathcal{G}^H_{\bullet,\mathcal{K}_f}(f)\mathcal{G}_{\bullet,\mathcal{K}_f}(f)$$

as a function of frequency. Note that the discontinuities in these plots are expected because $\mathcal{K}_f(f)$ is piecewise constant with discontinuities at the cell boundaries, *i.e.*, at $f = \gamma_1 = 0.15$ in this case. A numerical calculation yields the following frame bounds for the MIMO sampling scheme.

$$\begin{aligned} A &= \operatorname*{ess\,inf}_{f \in [0, \frac{1}{T}]} \lambda_{\min}(T\boldsymbol{S}(f)) = 0.1251, \\ B &= \operatorname*{ess\,sup}_{f \in [0, \frac{1}{T}]} \lambda_{\max}(T\boldsymbol{S}(f)) = 1.1105. \end{aligned}$$

Hence the condition number is $\sqrt{B/A} = 2.9790$. The other rank condition in (39) which needs to be verified at cell boundaries, also holds. Now, Theorem 2 guarantees the existence of a continuous filter matrix $\boldsymbol{H}(f)$ that achieves perfect reconstruction of the MIMO channel inputs.

The proof of Theorem 2 also provides, in principle, a method to construct a continuous reconstruction filter matrix $\boldsymbol{H}(f)$ when the conditions for its existence are satisfied. Specifically, fix $\mathcal{H}(f)$ at the boundary points per (46) (47), and then find a continuos solution $\mathcal{H}(f)$ to the systems of linear equations (48) and (49) for $f \in [\gamma_m, \gamma_{m+1}]$, $m \in \mathcal{M}$. The solution to these equations is in general nonunique, and a particular solution can be selected using additional criteria (for examples of

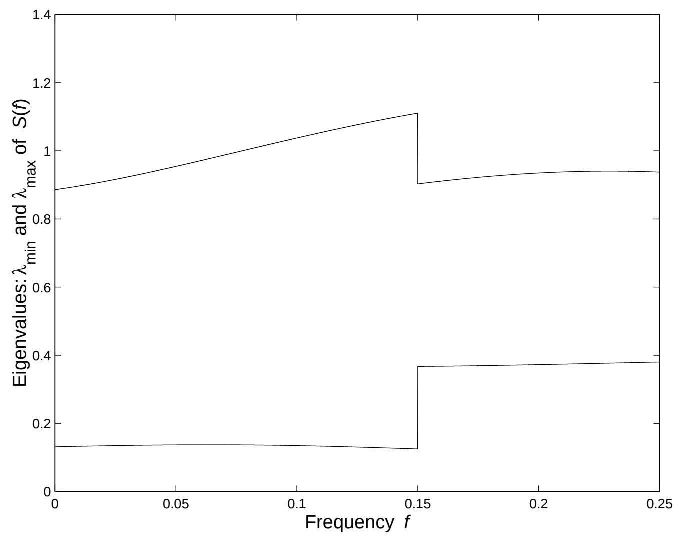

Fig. 4. Smallest and largest eigenvalues of $\boldsymbol{S}(f)$.

such designs in the single channel case, see [31]). For example, the minimum norm solution will lead to minimum amplification of additive white noise on the sampled signals (due to e.g., quantization error). In any event, the final filter matrix $\boldsymbol{H}(f)$ is obtained from $\boldsymbol{\mathcal{H}}(f)$ via (14).

V. Conclusion

In this paper, we studied the uniform MIMO sampling problem. This scheme encompasses periodic nonuniform multicoset sampling, Papoulis' generalized sampling, and vector sampling schemes as a special cases. The MIMO problem is motivated by the problem of channel equalization from the sampled channel outputs. We presented necessary and sufficient conditions for perfect reconstruction of the signals, or equivalently, perfect inversion of the channel, when the input signals lie in the space of multiband signals with different band structures. We also presented the appropriate conditions for the existence of reconstruction filters with continuous frequency responses, and specified them as solutions to a system of linear equations. The continuity property is important for the implementation of the reconstruction system because continuity allows arbitrary close $\mathcal{H}^\infty$ approximation of the filter responses by sufficiently long FIR filters. We address the problem of reconstruction filter design using FIR filters in [32]. Finally, we demonstrate that perfect reconstruction of the inputs is possible even when the channel outputs are sampled at a total combined rate that is smaller than the individual Nyquist rates for the inputs. In an example, we showed how to achieve the lower bound on the sampling density derived in [29] by multicoset sampling, however the question whether these lower bounds are generally achievable remains open.

Appendix

Proof of Lemma 1

Observe that $\boldsymbol{C}^H(f)\boldsymbol{C}(f)$ is non-singular for all $f \in [\alpha, \beta]$ because $\operatorname{rank} \boldsymbol{C}(f) = q$. In fact

$$\boldsymbol{C}^H(f)\boldsymbol{C}(f) > \epsilon^2 \boldsymbol{I}_q, \quad f \in [\alpha, \beta],$$

where ϵ is the minimum value of the smallest singular value of $\boldsymbol{C}(f)$ on $[\alpha, \beta]$:

$$\epsilon = \inf_{f \in [\alpha,\beta]} \sigma_{\min}(\boldsymbol{C}(f)) = \min_{f \in [\alpha,\beta]} \sigma_{\min}(\boldsymbol{C}(f)) > 0.$$

This is because $\boldsymbol{C}(f)$ is continuous, implying that $\sigma_{\min}(\boldsymbol{C}(f))$, which is also continuous on the compact set $[\alpha, \beta]$, attains its infimum. Therefore

$$\boldsymbol{C}^\dagger(f) = \left(\boldsymbol{C}^H(f)\boldsymbol{C}(f)\right)^{-1}\boldsymbol{C}^H(f),$$
$$\boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(f))} = \boldsymbol{C}(f)\left(\boldsymbol{C}^H(f)\boldsymbol{C}(f)\right)^{-1}\boldsymbol{C}^H(f),$$

are also a continuous functions of f. Note that $\boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(f))}$ is the orthogonal projection onto the range space of $\boldsymbol{C}(f)$. Define

$$\boldsymbol{E}_1 := \boldsymbol{E}_\alpha - \boldsymbol{D}(\alpha)\boldsymbol{C}^\dagger(\alpha),$$
$$\boldsymbol{E}_2 := \boldsymbol{E}_\beta - \boldsymbol{D}(\beta)\boldsymbol{C}^\dagger(\beta).$$

It follows that $\boldsymbol{E}_1\boldsymbol{C}(\alpha) = \boldsymbol{E}_2\boldsymbol{C}(\beta) = \boldsymbol{0}$, implying that

$$\boldsymbol{E}_1\boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(\alpha))} = \boldsymbol{E}_2\boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(\beta))} = \boldsymbol{0}. \tag{A.1}$$

Now take

$$\boldsymbol{E}(f) := \boldsymbol{D}(f)\boldsymbol{C}^\dagger(f) + \left(\frac{(f-\alpha)}{(\beta-\alpha)}\boldsymbol{E}_2 + \frac{(\beta-f)}{(\beta-\alpha)}\boldsymbol{E}_1\right)$$
$$\times \left(\boldsymbol{I}_q - \boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(f))}\right).$$

This is a valid solution because it is continuous and meets the requirements:

$$\boldsymbol{E}(f)\boldsymbol{C}(f) = \boldsymbol{D}(f) + \left(\frac{(f-\alpha)}{(\beta-\alpha)}\boldsymbol{E}_2 + \frac{(\beta-f)}{(\beta-\alpha)}\boldsymbol{E}_1\right)$$
$$\times \left(\boldsymbol{C}(f) - \boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(f))}\boldsymbol{C}(f)\right) = \boldsymbol{D}(f),$$
$$\boldsymbol{E}(\alpha) = \boldsymbol{D}(\alpha)\boldsymbol{C}^\dagger(\alpha) + \boldsymbol{E}_1 - \boldsymbol{E}_1\boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(\alpha))} = \boldsymbol{E}_\alpha,$$
$$\boldsymbol{E}(\beta) = \boldsymbol{D}(\beta)\boldsymbol{C}^\dagger(\beta) + \boldsymbol{E}_2 - \boldsymbol{E}_2\boldsymbol{P}_{\mathcal{R}(\boldsymbol{C}(\beta))} = \boldsymbol{E}_\beta.$$

The last two equations follow from (A.1). ■

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