EE 4078 DIGITAL SIGNAL PROCESSING

Prof. J. H. McClellan
Georgia Tech
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- FIR DESIGN via WINDOWING
- KAISER WINDOW EXAMPLES
- DESIGN FORMULA RELATES FILTER LENGTH TO RIPPLE HEIGHT
- OPTIMAL CHEBYSHEV APPROXIMATION for FIR DESIGN
- REMEZ EXCHANGE ALGORITHM
- EXAMPLES of EQUIRIPPLE FILTERS
STEPS in WINDOW FILTER DESIGN

1. Create the ideal impulse response, using the inverse DTFT to obtain \( h_d[n] \):

\[
h_d[n] = \frac{1}{\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} \, d\omega
\]

where \( H_d(e^{j\omega}) \) is the ideal frequency response. For example,

\[
h_d[n] = \frac{\sin \omega_c n}{\pi n}
\]

is an infinitely long “sinc” function, \(-\infty < n < \infty\).

2. NOTE: If the length of the window is \( L \), then the “ideal” frequency response must contain a linear phase term. For example, the ideal LPF would be specified as:

\[
H_d(e^{j\omega}) = \begin{cases} 
1 \cdot e^{-j\omega(L-1)/2} & -\omega_c \leq \omega \leq +\omega_c \\
0 & \omega_c \leq |\omega| < \pi
\end{cases}
\]

This allows both even length and odd length filters to be designed.

3. Create the FIR filter coefficients by multiplying by the window:

\[
h[n] = w[n] \cdot h_d[n] \quad n = 0, 1, \ldots, L-1
\]
4. In the frequency domain, this windowing operation results in a convolution of the ideal frequency response with the Fourier transform of the window, $W(e^{j\omega})$.

$$H(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta}) W(e^{j(\omega - \theta)}) \, d\theta$$

Note that this convolution is periodic with period $2\pi$.

5. *Transition Width*: The result is that the ideal frequency response is smeared by the convolution, so the actual frequency response has a smooth roll off from the passband to the stopband.

6. *Passband and Stopband Deviations*: In addition, all windows have sidelobes in their Fourier transforms, so the convolution gives rise to ripples in the frequency response of the FIR filter.
The Kaiser window of length $L$ is based on the modified Bessel function $I_0(x)$:

$$w[n] = \frac{I_0(\beta \sqrt{1 - (n - M)^2/M^2})}{I_0(\beta)} \quad \text{for } n = 0, 1, \ldots, L-1$$

The mid-point $M$ is $M = \frac{1}{2}(L-1)$; so, for an odd-length window, $M$ is an integer. The parameter $\beta$ should be chosen between 0 and 10 for useful windows.

The relationship between $\beta$ and the ripple height in the stopband (or passband) is:

$$\beta = \begin{cases} 
0 & \text{ATT} < 21 \\
0.5842(\text{ATT} - 21)^{0.4} + 0.07886(\text{ATT} - 21) & 21 \leq \text{ATT} \leq 50 \\
0.1102(\text{ATT} - 8.7) & 50 < \text{ATT}
\end{cases}$$

where $\text{ATT} = -20 \log_{10}(\delta_s)$ is the ripple height in dB.
WINDOWS & TRANSFORMS for $L = 49$

WINDOWS: Hamming, Hann, & Triangle

INDEX (n)

Amplitude

Triangular
Hamming
Hann

KAISER WINDOWS: $\beta = 2, 5, 8$

INDEX (n)

Amplitude

Kaiser(2.0)
Kaiser(5.0)
Kaiser(8.0)

Fourier Transform of Windows

Normalized Frequency ($f_s = 1$)

Log Magnitude

Rectangular
Hamming
Hann

Linear Magnitude

Kaiser(2.0)
Kaiser(5.0)
Kaiser(8.0)
WINDOWED FILTER DESIGN (L = 49)

**Ideal Impulse Response**
- Amplitude vs. Index (n)
- Hamming Window (scaled by 2*fc)

**Hamming Windowed Impulse Response**
- Amplitude vs. Index (n)
- Hamming Window (scaled by 2*fc)

**LPF with cutoff 0.05 via windows**
- Linear Magnitude
- Log Magnitude
- Rectangular, Hamming, Triangular (dotted)

Rectangular (dashed) shows Gibbs’ Effect.
WINDOW (L = 49) vs. REMEZ FILTER DESIGN

LPF with cutoff 0.05 via windows

- Rectangular
- Triangular (dotted)
- Hamming

Normalized Frequency (f_s = 1)

Log Magnitude

0 -100 -90 -80 -70 -60 -50 -40 -30 -20 -10 0

0 0.05 0.1 0.15 0.2

LPF with cutoff 0.05 via windows

- Parks-McClellan (solid)
  passband = 2%, stopband = 8%
  of the sampling frequency
- Kaiser (5.0) (dotted)
- Hamming (dashed)

Normalized Frequency (f_s = 1)

Linear Magnitude

0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

0 0.05 0.1 0.15 0.2

LPF with cutoff 0.05 via windows

- Kaiser(2.0)
- Kaiser(5.0)
- Kaiser(8.0)

Normalized Frequency (f_s = 1)

Log Magnitude

0 -100 -90 -80 -70 -60 -50 -40 -30 -20 -10 0

0 0.05 0.1 0.15 0.2

LPF with cutoff 0.05 via windows

- Parks-McClellan (equiripple)
- Hamming
- Kaiser (5.0)

Normalized Frequency (f_s = 1)

Log Magnitude

0 -100 -90 -80 -70 -60 -50 -40 -30 -20 -10 0

0 0.05 0.1 0.15 0.2
1. The desired frequency response can be written as

\[ H_d(e^{j\omega}) = A(e^{j\omega}) e^{-j(\alpha\omega + \beta)} \]

where \( \alpha = (L - 1)/2 \) always, and \( \beta = 0 \) for filters with even symmetry. Since \( A(e^{j\omega}) \) is a real-valued function, the Chebyshev approximation is applied to \( A(e^{j\omega}) \) and the linear phase comes for free.

2. The mathematical theory of *Chebyshev Approximation* is applied. In this type of optimization, the *maximum value of the error is minimized*, as opposed to the error energy as in least-squares.

3. Minimizing the max error is consistent with the desire to keep the passband and stopband deviations as small as possible. (Recall that least-squares suffers from the Gibbs’ effect).

4. Minimization of the max error does not permit the use of derivatives to find the optimal solution.

5. The *Alternation Theorem* gives the necessary and sufficient conditions for the optimum in terms of equal-height ripples in the (weighted) error function.

6. The *Remez exchange algorithm* will compute the optimal approximation by searching for the locations of the peaks in the error function. This algorithm is iterative.

7. The inputs to the algorithm are the filter length \( L \), the locations of the passband and stopband cutoff frequencies: \( \omega_p \) and \( \omega_s \), and a weight function to weight the error in the passband and stopband differently.

8. *Transition Width*: is minimized among all FIR filters with the same deviations.

9. *Passband and Stopband Deviations*: The response is equi-ripple, it does not fall off away from the transition region.
## FOUR TYPES of LINEAR PHASE FIR FILTERS

<table>
<thead>
<tr>
<th>Even Symmetry</th>
<th>Odd Length ($L$)</th>
<th>Even Length ($L$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(\alpha + n) = h(\alpha - n)$</td>
<td>$\frac{1}{2}[L-1] \sum_{k=0}^{a} a(k) \cos(\omega k)$</td>
<td>$\frac{1}{2}L \sum_{k=1}^{b} b(k) \cos(\omega [k - \frac{1}{2}])$</td>
</tr>
<tr>
<td>$\alpha = \frac{L-1}{2}$</td>
<td>$a(0) = h(\frac{L-1}{2})$</td>
<td>zero at $\omega = \pi$</td>
</tr>
<tr>
<td>$\beta = 0$</td>
<td>$a(k) = 2h(\frac{L-1}{2} - k)$</td>
<td>$b(k) = 2h(\frac{L}{2} - k)$</td>
</tr>
</tbody>
</table>

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<tr>
<th>Odd Symmetry</th>
<th>TYPE III</th>
<th>TYPE IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(\alpha + n) = -h(\alpha - n)$</td>
<td>$\frac{1}{2}[L-1] \sum_{k=1}^{c} c(k) \sin(\omega k)$</td>
<td>$\frac{1}{2}L \sum_{k=1}^{d} d(k) \sin(\omega [k - \frac{1}{2}])$</td>
</tr>
<tr>
<td>$\alpha = \frac{L-1}{2}$</td>
<td>$\text{zeros at } \omega = 0, \pi$</td>
<td>zero at $\omega = 0$</td>
</tr>
<tr>
<td>$\beta = \frac{\pi}{2}$</td>
<td>$c(k) = 2h(\frac{L-1}{2} - k)$</td>
<td>$d(k) = 2h(\frac{L}{2} - k)$</td>
</tr>
<tr>
<td>$h(\frac{L-1}{2}) = 0$</td>
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<tr>
<td>$\sin(\omega) \sum_{k=0}^{\alpha-1} \hat{c}(k) \cos(\omega k)$</td>
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</tr>
<tr>
<td>$\sin(\frac{1}{2}\omega) \sum_{k=0}^{\frac{L}{2}-1} \hat{d}(k) \cos(\omega k)$</td>
<td>$</td>
<td>\sin(\omega) \sum_{k=0}^{\alpha-1} \hat{c}(k) \cos(\omega k)$</td>
</tr>
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</table>
The general form of the system function for a causal FIR filter is:

$$H(z) = \sum_{n=0}^{L-1} h(n)z^{-n}$$

where $L$ is the length of the impulse response. If the system has “generalized linear phase” then the frequency response can always be written in the form:

$$H(e^{j\omega}) = e^{j(\beta-\omega)\alpha} \cdot A(\omega)$$

where $A(\omega)$ is a purely-real function that may be positive or negative. The constant $\alpha$ is always equal to $\frac{1}{2}(L - 1)$. There are exactly four types of linear phase FIR filters, depending on the length of the impulse response and the symmetry of the impulse response.

I. Odd length: $L$, even symmetry: $h(n) = h(L - 1 - n)$. In this case, $\beta = 0$ and $\alpha = \frac{1}{2}(L - 1)$. Since the impulse response is symmetric around the point $n = \alpha$, the real-valued function $A(\omega)$ can be derived by collecting terms pairwise in the sum for $H(e^{j\omega})$:

$$H(e^{j\omega}) = e^{-j\alpha\omega} \cdot \left[ h(0)e^{j\omega} + h(L - 1)e^{-j\omega} + h(1)e^{j(\alpha-1)\omega} + h(L - 2)e^{-j(\alpha-1)\omega} + \ldots \right] + h(\frac{1}{2}[L - 1] - 1)e^{j\omega} + h(\frac{1}{2}[L - 1] + 1)e^{-j\omega} + h(\frac{1}{2}[L - 1])$$

The symmetry condition, $h(\alpha + n) = h(\alpha - n)$, can be invoked to write:

$$H(e^{j\omega}) = e^{-j\alpha\omega} \cdot \left[ h(0)(e^{j\omega} + e^{-j\omega}) + h(1)(e^{j(\alpha-1)\omega} + e^{-j(\alpha-1)\omega}) + \ldots \right] + h(\frac{1}{2}[L - 1] - 1)(e^{j\omega} + e^{-j\omega}) + h(\frac{1}{2}[L - 1])$$

Finally, each pair of terms can be reduced to a purely-real term; namely a cosine function:

$$A(\omega) = \sum_{k=0}^{\frac{1}{2}[L-1]} a(k) \cos(k\omega)$$

where the coefficients $a(k)$ are

$$a(k) = \begin{cases} 
    h(\alpha) & \text{for } k = 0; \\
    2 \cdot h(\alpha - k) & \text{for } k = 1, 2, \ldots \alpha \\
    \end{cases} \quad (\alpha = \frac{1}{2}(L - 1))$$
ALTERNATION THEOREM for
OPTIMAL CHEBYSHEV APPROXIMATION

Necessary and Sufficient Conditions

If \( P(e^{j\omega}) \) is a linear combination of \( v \) cosine functions:

\[
P(e^{j\omega}) = \sum_{k=0}^{v-1} \beta_k \cos(\omega k)
\]

then a necessary and sufficient condition that \( P(e^{j\omega}) \) be the unique best weighted Chebyshev approximation to a continuous function \( D(e^{j\omega}) \) over a subset \( \tilde{\Omega} \) of the interval \([0, \pi]\) is that the weighted error function:

\[
E(e^{j\omega}) = W(e^{j\omega}) \{ D(e^{j\omega}) - P(e^{j\omega}) \}
\]

exhibit at least \( v \) “alternations” within \( \tilde{\Omega} \). That is, there must be at least \( v + 1 \) distinct points, \( \omega_i \in \tilde{\Omega} \),

\[
\omega_1 < \omega_2 < \cdots < \omega_{v+1}
\]

where the weighted error attains its maximum value at each \( \omega_i \):

\[
|E(e^{j\omega_i})| = \max_{\tilde{\Omega}} |E(e^{j\omega})| = \| E(e^{j\omega}) \|_\infty
\]

AND the weighted error alternates sign at each successive point:

\[
E(e^{j\omega_{i+1}}) = -E(e^{j\omega_i}) \quad i = 1, 2, \ldots, v
\]

These \( v + 1 \) points are called extremal frequencies.

NOTE(1): there may be more than \( v + 1 \) extremal frequencies; the theorem only sets a minimum.

NOTE(2): the weight function, \( W(e^{j\omega}) \), must be strictly positive and continuous over \( \tilde{\Omega} \).
REMEZ (Parks-McClellan) FILTER DESIGN EXAMPLES

Fourier Transform of PMFIR Design  $L=15$

$\# \text{ Extremal Freqs} = 9$

$\delta_{\text{pass}} = 0.13675$
$\delta_{\text{stop}} = 0.068377$

passband cutoff = 0.19531
stopband cutoff = 0.25391

Fourier Transform of Non-Optimal Design  $L=15$

Possible Extrema = 10

$\delta_{\text{stop}} = 0.039209$
$\omega_{\text{pass}} = 0.22656$
$\omega_{\text{stop}} = 0.27344$

Fourier Transform of PMFIR Design  $L=15$

$\# \text{ Extremal Freqs} = 10$

(MAXIMAL NUMBER)

$\delta_{\text{stop}} = 0.1086$
$\omega_{\text{pass}} = 0.22656$
$\omega_{\text{stop}} = 0.27344$

PMFIR IMPULSE RESPONSE  $L=15$

SINC FUNCTION ENVELOPE
OPERATION of REMEZ EXCHANGE ALGORITHM (Parks-McClellan)

1. INITIAL GUESS of \( n + 1 \) EXTREMAL FREQUENCIES
2. CALCULATE THE OPTIMUM \( \delta \) ON EXTREMAL SET
3. INTERPOLATE THRU \( n + 1 \) POINTS TO OBTAIN \( A(\omega) \)
4. CALCULATE ERROR \( E(\omega) \) & FIND LOCAL MAXIMA WHERE \( |E(\omega)| \geq \delta \)
5. MORE THAN \( n + 1 \) EXTREMA?
   - YES: RETAIN \( n + 1 \) LARGEST EXTREMA
   - NO: CHECK WHETHER THE EXTREMAL POINTS CHANGED
     - CHANGED: RETAIN \( n + 1 \) LARGEST EXTREMA
     - UNCHANGED: BEST APPROXIMATION !!
**DESIGN FORMULAE for LINEAR PHASE FIR FILTERS**

**REMEZ** OPTIMAL EQUIRIPPLE FIR LPF:

\[(L - 1)\Delta\omega \approx \frac{-20\log_{10}\sqrt{\delta_p\delta_s} - 13}{2.324}\]

**KAISER** WINDOW FIR LPF:

\[(L - 1)\Delta\omega \approx \frac{-20\log_{10}\delta - 8}{2.285}\]

NOTE: for Kaiser window designs, ripples must be the same: \(\delta = \delta_p = \delta_s\).

The smaller the value of \((L - 1)\Delta\omega\), the better the filter.

The Transition Width is: \(\Delta\omega \overset{\text{def}}{=} \omega_s - \omega_p\)

and the Filter Length is: \(L\)

**EXAMPLE:**

\[
\text{ATT}_{\text{PMFIR}} = 2.324(L - 1)\Delta\omega + 13
\]

\[
\text{ATT}_{\text{Kaiser}} = 2.285(L - 1)\Delta\omega + 8
\]

So there is about a 5-dB difference in attenuation when the same specs are used for the other filter design parameters \((L\) and \(\Delta\omega\)).