

## Useful Circuit Theorems

### Impedance of a Two-Terminal $RC$ Network

Consider any two-terminal  $RC$  network. If the impedance at dc is not infinite, the impedance can be written

$$Z = R_{dc} \frac{1 + \tau_z s}{1 + \tau_p s}$$

where  $R_{dc}$  is the dc resistance of the circuit,  $\tau_o$  is its open-circuit time constant, and  $\tau_z$  is its short-circuit time constant.

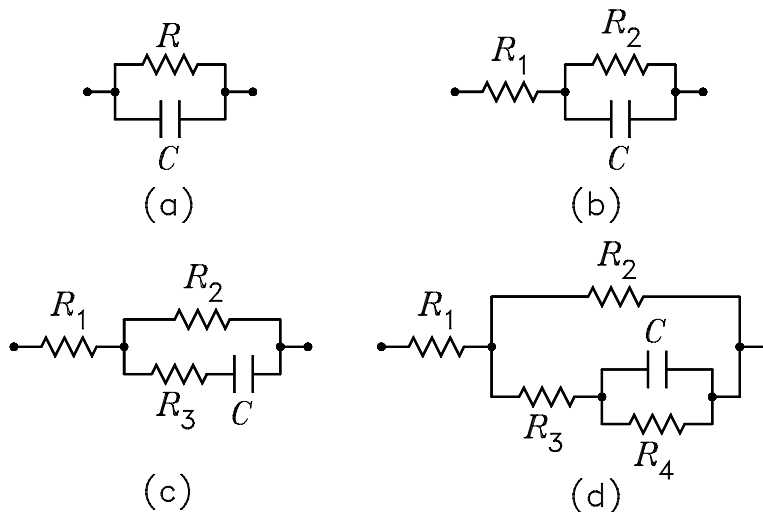


Figure 1: Example two-terminal circuits.

For example applications of the theorem, consider the circuits shown in Fig. 1. In order from (a) to (d), the impedances are given by

$$Z = R \frac{1}{1 + RCs}$$

$$Z = (R_1 + R_2) \frac{1 + R_1 \parallel R_2 Cs}{1 + R_2 Cs}$$

$$Z = (R_1 + R_2) \frac{1 + (R_1 \parallel R_3 + R_2) Cs}{1 + (R_2 + R_3) Cs}$$

$$Z = [R_1 + R_2 \parallel (R_3 + R_4)] \frac{1 + (R_1 \parallel R_2 + R_3) \parallel R_4 Cs}{1 + (R_2 + R_3) \parallel R_4 Cs}$$

Although the theorem is strictly valid for circuits containing only one capacitor, it can be applied to circuits containing more than one capacitor if the adjacent poles and zeroes in the transfer function are well removed, preferably by a decade or more. Consider the circuit shown in Fig. 2. Let us assume that  $C_2$  and  $C_3$  are open circuits in the frequency range in which  $C_1$  is active,  $C_1$  is a short circuit and  $C_3$  is an open circuit in the frequency range in which  $C_2$  is active, and  $C_1$  and  $C_2$  are short circuits in the frequency range in which  $C_3$  becomes active.

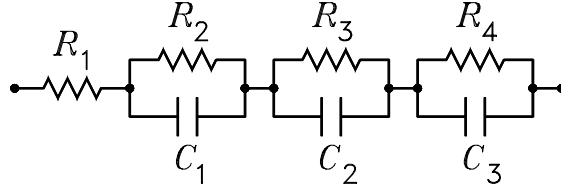


Figure 2: Example circuit containing more than one capacitor.

With the above information, the impedance in the range where  $C_2$  and  $C_3$  are open circuits is given by

$$\begin{aligned} Z_1 &= (R_1 + R_2 + R_3 + R_4) \frac{1 + R_2 \parallel (R_1 + R_3 + R_4) C_1 s}{1 + R_2 C_1 s} \\ &= (R_1 + R_2 + R_3 + R_4) \frac{1 + s/\omega_2}{1 + s/\omega_1} \end{aligned}$$

where

$$\omega_1 = \frac{1}{R_2 C_1} \quad \omega_2 = \frac{1}{1 + R_2 \parallel (R_1 + R_3 + R_4) C_1}$$

At low frequencies,  $Z_1$  starts at the value  $R_1 + R_2 + R_3 + R_4$  and shelves at high frequencies at the value  $R_1 + R_3 + R_4$ . The impedance in the range where  $C_1$  is a short circuit and  $C_3$  is an open circuit is given by

$$\begin{aligned} Z_2 &= (R_1 + R_3 + R_4) \frac{1 + R_3 \parallel (R_1 + R_4) C_2 s}{1 + R_3 C_2 s} \\ &= (R_1 + R_3 + R_4) \frac{1 + s/\omega_4}{1 + s/\omega_3} \end{aligned}$$

where

$$\omega_3 = \frac{1}{R_3 C_2} \quad \omega_4 = \frac{1}{R_3 \parallel (R_1 + R_4) C_2}$$

At low frequencies,  $Z_2$  starts at the value  $R_1 + R_3 + R_4$  and shelves at high frequencies at the value  $R_1 + R_4$ . The impedance in the range where  $C_1$  and  $C_2$  are short circuits is given by

$$\begin{aligned} Z_3 &= (R_1 + R_4) \frac{1 + R_4 \parallel R_1 C_3 s}{1 + R_4 C_3 s} \\ &= (R_1 + R_4) \frac{1 + s/\omega_6}{1 + s/\omega_5} \end{aligned}$$

where

$$\omega_5 = \frac{1}{R_4 C_3} \quad \omega_6 = \frac{1}{R_4 \parallel R_1 C_3}$$

At low frequencies,  $Z_3$  starts at the value  $R_1 + R_4$  and shelves at high frequencies at the value  $R_1$ .

The three impedances can be “pieced” together to obtain the overall impedance to obtain

$$Z = (R_1 + R_2 + R_3 + R_4) \frac{(1 + s/\omega_2)(1 + s/\omega_4)(1 + s/\omega_6)}{(1 + s/\omega_1)(1 + s/\omega_3)(1 + s/\omega_5)}$$

This expression is strictly if

$$\omega_1 \ll \omega_2 \ll \omega_3 \ll \omega_4 \ll \omega_5 \ll \omega_6$$

The straight-line approximation to the Bode magnitude plot for the impedance is shown in Fig. 3.

The impedance theorem can be used to write by inspection the transfer function of an inverting op-amp circuit where the input and feedback impedances contain no more than one capacitor each. Consider the circuit shown in Figure 4. The voltage gain can be written by inspection to obtain

$$\begin{aligned} \frac{V_2}{V_1} &= -\frac{R_3}{1 + R_3 C_2 s} \div \left[ (R_1 + R_2) \frac{1 + R_1 \parallel R_2 C_1 s}{1 + R_2 C_2 s} \right] \\ &= -\frac{R_3}{R_1 + R_2} \frac{1 + R_2 C_2 s}{(1 + R_1 \parallel R_2 C_1 s)(1 + R_3 C_2 s)} \end{aligned}$$

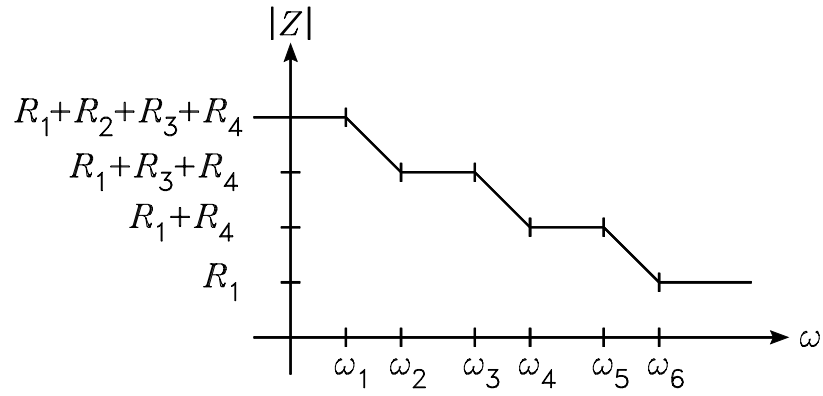


Figure 3: Bode magnitude plot for the impedance  $Z$ .

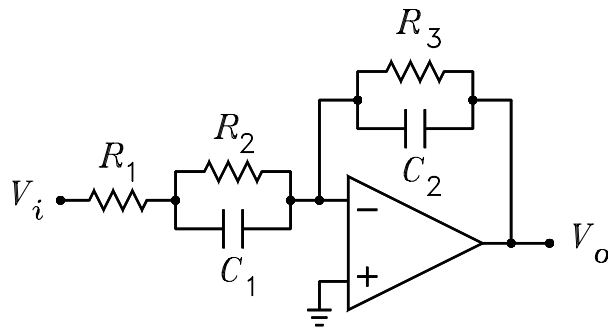


Figure 4: Inverting op-amp example.

## Impedance of an $RC$ Voltage-Divider Network

### Case 1 – Capacitor in Shunt Arm

Consider the voltage-divider network shown in Fig. 5(a). Let the impedance  $Z_2$  contain one capacitor and satisfies the condition for the two-terminal impedance theorem. By voltage division, the gain of the network is given by

$$\frac{V_2}{V_1} = \frac{Z_2}{R_1 + Z_2} = \frac{Z_2}{Z_3}$$

where  $Z_3 = R_1 + Z_2$ . The impedance  $Z_2$  can be written

$$Z_2 = R_2 \frac{1 + \tau_{2s}s}{1 + \tau_{2o}s}$$

where  $\tau_{2o}$  is the open-circuit time constant for  $Z_2$  and  $\tau_{2s}$  is its short-circuit time constant. The impedance  $Z_3$  can be written

$$Z_3 = (R_1 + R_2) \frac{1 + \tau_{3s}s}{1 + \tau_{3o}s}$$

where  $\tau_{3o}$  is the open-circuit time constant for  $Z_3$  and  $\tau_{3s}$  is its short-circuit time constant.

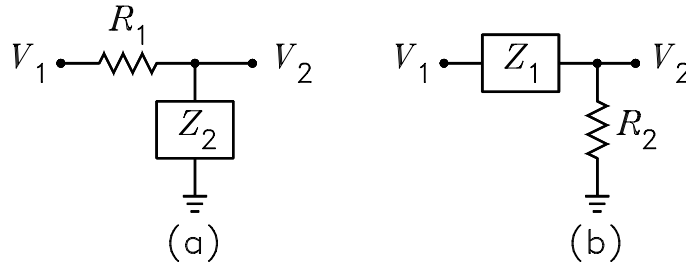


Figure 5: Voltage divider networks containing only one capacitor.

But the open-circuit time constants for  $Z_2$  and  $Z_3$  are the same. Let this be denoted by  $\tau_o = \tau_{2o} = \tau_{3o}$ . thus the two impedances can be written

$$Z_2 = R_2 \frac{1 + \tau_{2s}s}{1 + \tau_o s}$$

and

$$Z_3 = (R_1 + R_2) \frac{1 + \tau_{3s}s}{1 + \tau_o s}$$

It follows that the gain of the voltage divider is given by

$$\frac{V_2}{V_1} = \frac{R_2}{R_1 + R_2} \frac{1 + \tau_{2s}s}{1 + \tau_{3s}s}$$

Notice that the term  $1 + \tau_o s$  is canceled. Note also that the gain constant is the circuit gain at dc, the pole time constant  $\tau_{3s}$  is the time constant calculated with  $V_i = 0$ , and the zero time constant  $\tau_{2s}$  is the time constant with  $V_o = 0$ .

### Case 2 – Capacitor in Series Arm

Now consider the case shown in Fig. 5(b) where  $Z_1$  contains one capacitor and satisfies the condition for the two-terminal impedance theorem. By voltage division, the gain of the network is given by

$$\frac{V_2}{V_1} = \frac{R_2}{Z_1 + R_2} = \frac{R_2}{Z_3}$$

where  $Z_3 = Z_1 + R_2$ . The impedance  $Z_3$  can be written

$$Z_3 = (R_1 + R_2) \frac{1 + \tau_{3s}s}{1 + \tau_{3o}s}$$

where  $\tau_{3o}$  is the open-circuit time constant for  $Z_3$  and  $\tau_{3s}$  is its short-circuit time constant.

It follows that the gain of the voltage divider is given by

$$\frac{V_2}{V_1} = \frac{R_2}{R_1 + R_2} \frac{1 + \tau_{3o}s}{1 + \tau_{3s}s}$$

But the open-circuit time constant for  $Z_3$  is equal to the open-circuit time constant for  $Z_1$ , i.e.  $\tau_{3o} = \tau_{1o}$ . Thus the gain of the circuit can be written

$$\frac{V_2}{V_1} = \frac{R_2}{R_1 + R_2} \frac{1 + \tau_{1o}s}{1 + \tau_{3s}s}$$

Note that the gain constant is the circuit gain at dc, the pole time constant  $\tau_{3s}$  is the time constant calculated with  $V_i = 0$ , and the zero time constant  $\tau_{1s}$  is the time constant with the  $V_i$  node floating, i.e. open circuited.

### Combined Theorem

We seek to combine the two theorems into one which gives the correct answer for both cases. In the second case, the time constant  $\tau_{1o}$  is the same as the time constant calculated with  $V_o = 0$ . Thus it follows that the two theorems can be combined to obtain the general solution

$$\frac{V_2}{V_1} = k_{dc} \frac{1 + \tau_2s}{1 + \tau_1s}$$

where  $k_{dc}$  is the dc gain,  $\tau_1$  is the time constant with  $V_1 = 0$ , and  $\tau_2$  is the time constant with  $V_2 = 0$ .

As an example, consider the circuit shown in Fig. 6(a). By inspection, the voltage gain can be written

$$\frac{V_2}{V_1} = \frac{R_2 + R_3}{R_1 + R_2 + R_3} \frac{1 + (R_2 \parallel R_3 + R_4)Cs}{1 + [(R_1 + R_2) \parallel R_3 + R_4]Cs}$$

The Bode magnitude plot is that of a low-pass shelving function.

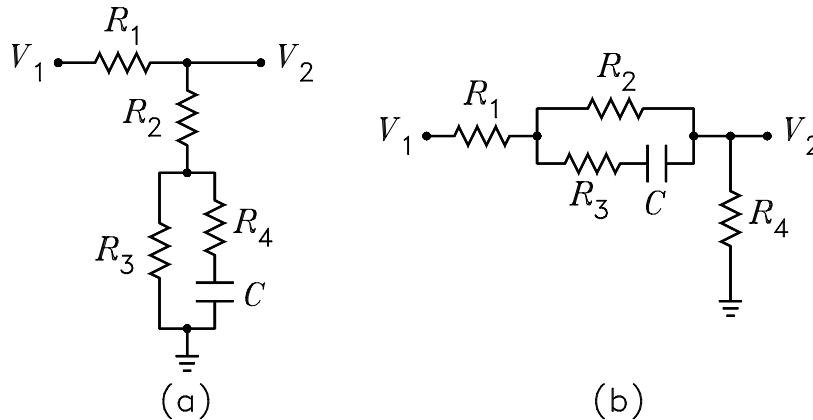


Figure 6: Example circuits for the voltage-divider theorem.

The voltage gain of the circuit in Fig. 6(b) is given by

$$\frac{V_2}{V_1} = \frac{R_4}{R_1 + R_2 + R_4} \frac{1 + (R_2 + R_3)Cs}{1 + [(R_1 + R_4) \parallel R_2 + R_3]Cs}$$

The Bode magnitude plot is that of a high-pass shelving function.

The voltage-divider theorem can be used to write the voltage gain expression for a non-inverting op-amp circuit. Consider the circuit shown in Fig. 7. The voltage divider network in Fig. 6(a) is shown connected between the output of the op amp and its inverting input. Because the op amp has negative feedback, there

is a virtual short between its + and - inputs. Thus the voltage output of the voltage divider is  $V_2 = V_i$  and its voltage input is  $V_1 = V_o$ . It follows that the voltage gain of the circuit is given by

$$\frac{V_o}{V_i} = \frac{V_1}{V_2} = \left(\frac{V_2}{V_1}\right)^{-1} = \frac{R_1 + R_2 + R_4}{R_4} \frac{1 + [(R_1 + R_4) \parallel (R_2 + R_3)]Cs}{1 + (R_2 + R_3)Cs}$$

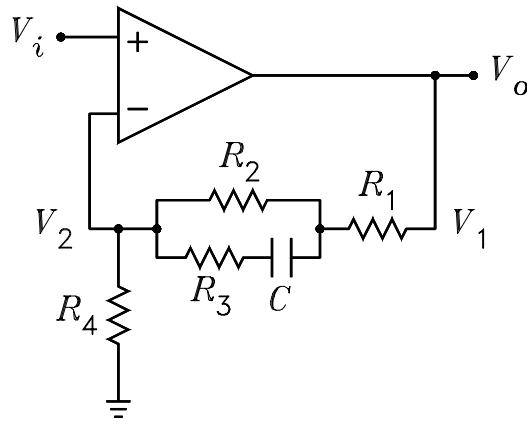


Figure 7: Non-inverting op-amp example.