

GEORGIA INSTITUTE OF TECHNOLOGY  
School of Electrical and Computer Engineering

Course ECE 2040  
Circuit Analysis

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**Problem Set #7–Solutions**

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**Problem 7.1:** This problem derives a number of identities concerning the complex conjugate.

(a) Using either the rectangular or polar representation for each complex variable, show that:

(i)  $(s_1 + s_2)^* = s_1^* + s_2^*$ .

(ii)  $(as_1)^* = as_1^*$ , where  $a$  is any real number.

(iii)  $(s_1s_2)^* = s_1^*s_2^*$ .

(iv)  $(e^s)^* = e^{s^*}$ .

(iv)  $(s^a)^* = (s^*)^a$ , where  $a$  is any real number.

(vi)  $\left(\frac{s_1}{s_2}\right)^* = \frac{s_1^*}{s_2^*}$

(b) Using the results that you derived in (a), show that

$$[P(s)]^* = P(s^*),$$

where  $P(s)$  is a polynomial of degree  $n$  in  $s$  with real coefficients

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0.$$

(c) Using the results of (a) and (b), show that

$$[Q(s)]^* = Q(s^*),$$

where  $Q(s)$  is the ratio of two polynomials with real coefficients

$$Q(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}.$$

(d) Using the results of (b), show that the complex roots of a polynomial with real coefficients occur in complex conjugate pairs, i.e., show that if  $P(s_1) = 0$ , where  $P(s)$  is a polynomial with real coefficients, then  $P(s_1^*) = 0$ .

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**Solution:**

(a) (i)

$$\begin{aligned} (s_1 + s_2)^* &= [\sigma_1 + j\omega_1 + \sigma_2 + j\omega_2]^* \\ &= [(\sigma_1 + \sigma_2) + j(\omega_1 + \omega_2)]^* \\ &= (\sigma_1 + \sigma_2) - j(\omega_1 + \omega_2) \\ &= (\sigma_1 - j\omega_1) + (\sigma_2 - j\omega_2) = s_1^* + s_2^* \end{aligned}$$

(ii)

$$\begin{aligned}(as_1)^* &= (a[\sigma_1 + j\omega_1])^* = (a\sigma_1 + ja\omega_1)^* \\ &= a\sigma_1 - ja\omega_1 = a(\sigma_1 - j\omega_1) = as_1^*\end{aligned}$$

(iii)

$$\begin{aligned}(s_1s_2)^* &= (|s_1|e^{j\arg s_1} \cdot |s_2|e^{j\arg s_2})^* \\ &= (|s_1| \cdot |s_2|e^{j[\arg s_1 + \arg s_2]})^* \\ &= |s_1| \cdot |s_2|e^{-j[\arg s_1 + \arg s_2]} \\ &= |s_1|e^{-j\arg s_1} \cdot |s_2|e^{-j\arg s_2} = s_1^*s_2^*\end{aligned}$$

(iv)

$$\begin{aligned}(e^s)^* &= \left[ e^{\Re(s) + j\Im(s)} \right]^* \left[ e^{\Re(s)} e^{j\Im(s)} \right]^* \\ &= e^{\Re(s)} e^{-j\Im(s)} = e^{\Re(s) - j\Im(s)} = e^{s^*}\end{aligned}$$

(v)

$$\begin{aligned}(s^a)^* &= \left[ |s|^a e^{-ja\arg(s)} \right]^* = |s|^a e^{-ja\arg(s)} \\ &= \left[ |s| e^{-j\arg(s)} \right]^a = (s^*)^a\end{aligned}$$

(vi)

$$\begin{aligned}\left(\frac{s_1}{s_2}\right)^* &= \left(\frac{|s_1|e^{j\arg s_1}}{|s_2|e^{j\arg s_2}}\right)^* = \left(\frac{|s_1|}{|s_2|}e^{j(\arg(s_1) - \arg(s_2))}\right)^* \\ &= \frac{|s_1|}{|s_2|}e^{-j(\arg(s_1) - \arg(s_2))} \\ &= \frac{|s_1|e^{-j\arg(s_1)}}{|s_2|e^{-j\arg(s_2)}} = \frac{s_1^*}{s_2^*}\end{aligned}$$

(b)

$$\begin{aligned}[P(s)]^* &= [a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0]^* \\ &= (a_n s^n)^* + (a_{n-1} s^{n-1})^* + \dots + (a_1 s)^* + (a_0)^* \\ &= a_n (s^n)^* + a_{n-1} (s^{n-1})^* + \dots + a_1 (s)^* + a_0 \\ &= a_n (s^*)^n + a_{n-1} (s^*)^{n-1} + \dots + a_1 s^* + a_0 = P(s^*)\end{aligned}$$

(c)

$$[Q(s)]^* = \left[ \frac{B(s)}{A(s)} \right]^* = \frac{B^*(s)}{A^*(s)} = \frac{B(s^*)}{A(s^*)}.$$

(d) If  $s_1$  is a root of  $P(s)$ , then

$$P(s_1) = 0.$$

Taking the complex conjugate of both sides of this equation gives

$$[P(s_1)]^* = 0,$$

but since the coefficients of  $P(s)$  are real, this means (from (b))

$$P(s_1^*) = 0$$

and that  $s_1^*$  is a root of  $P(s)$ .

**Problem 7.2:** Find the inverse Laplace transform of

$$X(s) = \frac{2s + 6}{s(s^2 + 3s + 2)}.$$

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**Solution:**

$$\begin{aligned} X(s) &= \frac{2s + 6}{s(s^2 + 3s + 2)} = \frac{2s + 6}{s(s + 1)(s + 2)} \\ &= \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2} \end{aligned}$$

We can evaluate  $A$ ,  $B$ , and  $C$  using

$$\begin{aligned} A &= \lim_{s \rightarrow 0} \frac{2s + 6}{(s + 1)(s + 2)} = 3 \\ B &= \lim_{s \rightarrow -1} \frac{2s + 6}{s(s + 2)} = -4 \\ C &= \lim_{s \rightarrow -2} \frac{2s + 6}{s(s + 1)} = 1 \end{aligned}$$

Therefore,

$$x(t) = 3 - 4e^{-t} + e^{-2t}, \quad t > 0.$$

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**Problem 7.3:** The inverse Laplace transform of

$$X(s) = \frac{s + 2}{(s + 2)^2 + 9}$$

has the form

$$x(t) = Ke^{-at} \cos(\omega_0 t + \phi).$$

Determine the values of  $K$ ,  $a$ ,  $\omega_0$ , and  $\phi$ .

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**Solution:** The partial fraction expansion has the form

$$X(s) = \frac{A}{s + 2 - j3} + \frac{A^*}{s + 2 + j3}.$$

To determine  $A$  we evaluate the limit

$$A = \lim_{s \rightarrow -2 + j3} \frac{s + 2}{s + 2 + j3} = \frac{j3}{j6} = \frac{1}{2}.$$

Therefore,

$$x(t) = \frac{1}{2}e^{-2t}e^{j3t} + \frac{1}{2}e^{-2t}e^{-j3t} = e^{-2t} \cos 3t.$$

Thus,

$$K = 1, \quad a = 2, \quad \omega_0 = 3, \quad \phi = 0.$$

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**Problem 7.4:** Find the inverse Laplace transform of

$$X(s) = \frac{s + 3}{s^3 + 3s^2 + 6s + 4}.$$

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**Solution:**

$$\begin{aligned} X(s) &= \frac{s + 3}{s^3 + 3s^2 + 6s + 4} = \frac{s + 3}{(s + 1)(s^2 + 2s + 4)} \\ &= \frac{s + 3}{(s + 1)(s + 1 - j\sqrt{3})(s + 1 + j\sqrt{3})} \\ &= \frac{A}{s + 1} + \frac{B}{s + 1 - j\sqrt{3}} + \frac{B^*}{s + 1 + j\sqrt{3}}. \end{aligned}$$

$$A = \lim_{s \rightarrow -1} \frac{s + 3}{s^2 + 2s + 4} = \frac{2}{3}$$

$$B = \lim_{s \rightarrow (-1 + j\sqrt{3})} \frac{s + 3}{(s + 1)(s + 1 + j\sqrt{3})} = -\frac{1}{3} - j\frac{1}{2\sqrt{3}}$$

Therefore,

$$\begin{aligned} x(t) &= \frac{2}{3}e^{-t} + \left(-\frac{1}{3} - j\frac{1}{2\sqrt{3}}\right)e^{-t}e^{j\sqrt{3}t} + \left(-\frac{1}{3} + j\frac{1}{2\sqrt{3}}\right)e^{-t}e^{-j\sqrt{3}t} \\ &= \frac{2}{3}e^{-t} - \frac{1}{3}e^{-t}(e^{j\sqrt{3}t} + e^{-j\sqrt{3}t}) + \frac{1}{j2\sqrt{3}}e^{-t}(e^{j\sqrt{3}t} - e^{-j\sqrt{3}t}) \\ &= \frac{2}{3}e^{-t} - \frac{2}{3}e^{-t}\cos\sqrt{3}t + \frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t, \quad t > 0 \end{aligned}$$


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**Problem 7.5:** Find the inverse Laplace transform of

$$X(s) = \frac{5s - 1}{s^3 - 3s - 2}.$$

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**Solution:**

$$\begin{aligned} X(s) &= \frac{5s - 1}{s^3 - 3s + 2} = \frac{5s - 1}{(s + 1)^2(s - 2)} \\ &= \frac{A}{s - 2} + \frac{B}{s + 1} + \frac{C}{(s + 1)^2} \end{aligned}$$

$$A = \lim_{s \rightarrow 2} \frac{5s - 1}{(s + 1)^2} = 1$$

$$B = \lim_{s \rightarrow -1} \frac{d}{ds} \left( \frac{5s - 1}{s - 2} \right) = \lim_{s \rightarrow -1} \frac{(s - 2)5 - (5s - 1)}{(s - 2)^2} = -1$$

$$C = \lim_{s \rightarrow -1} \frac{5s - 1}{s - 2} = 2$$

Therefore,

$$x(t) = e^{-2t} - e^{-t} + 2te^{-t} = e^{-2t} + (2t - 1)e^{-t}, \quad t > 0.$$

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**Problem 7.6:** Find the signal  $x(t)$  for  $t \geq 0$  if its Laplace transform is

$$X(s) = \frac{s^3}{(s+1)([s+1]^2+4)}.$$

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**Solution:**

$$X(s) = \frac{s^3}{(s+1)(s^2+2s+5)} = 1 + \frac{A}{s+1} + \frac{B}{s+1-j2} + \frac{B^*}{s+1+j2}$$

$$A = \lim_{s \rightarrow -1} \frac{s^3}{s^2+2s+5} = -\frac{1}{4}$$

$$B = \lim_{s \rightarrow -1+j2} \frac{s^3}{(s+1)(s+1+j2)} = -\frac{11}{8} - \frac{1/2}{j2}$$

Therefore,

$$\begin{aligned} x(t) &= \delta(t) - \frac{1}{4}e^{-t} + \left(-\frac{11}{8} + \frac{1/2}{j2}\right)e^{-t}e^{j2t} + \left(-\frac{11}{8} - \frac{1/2}{j2}\right)e^{-t}e^{-j2t} \\ &= \delta(t) - e^{-t} \left(\frac{1}{4} + \frac{11}{4} \cos 2t - \frac{1}{2} \sin 2t\right), \quad t \geq 0 \end{aligned}$$

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