GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

**Course ECE 2040**

**Circuit Analysis**

March 16, 2001

## Problem Set #8–Solutions

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**Problem 8.1:** In the circuit of Figure 1 the source voltage is $v_s(t) = e^{-2t}$ for $t > 0$ and $v_c(0) = 1\,\text{V}$.

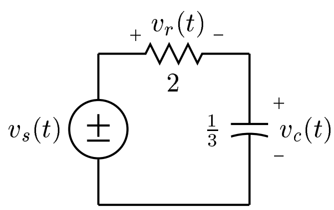


Figure 1: Circuit for Problem 8.1.

(a) Write the differential equation whose solution is the equilibrium solution for $v_c(t)$.

(b) Write the differential equation whose solution is the equilibrium solution for $v_r(t)$ and determine the value of $v_r(0)$.

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**Solution:**

(a) We can begin with a statement of KVL.

$$v_r(t) + v_c(t) = v_s(t).$$

We know, however, that

$$\begin{aligned} v_r(t) &= 2i_r(t) \\ &= 2i_c(t) \\ &= 2 \cdot \frac{1}{3}\frac{dv_c(t)}{dt}. \end{aligned}$$

Therefore,

$$\frac{2}{3}\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

or

$$\frac{dv_c(t)}{dt} + \frac{3}{2}v_c(t) = \frac{3}{2}v_s(t) = \frac{3}{2}e^{-2t}, \quad t > 0.$$

(b) Since $v_c(t) = v_s(t) - v_r(t)$, we can substitute into the solution of part (a).

$$\frac{dv_s(t)}{dt} - \frac{dv_r(t)}{dt} + \frac{3}{2}v_s(t) - \frac{3}{2}v_r(t) = \frac{3}{2}v_s(t).$$

Substituting for the known input and simplifying gives

$$\frac{dv_r(t)}{dt} + \frac{3}{2}v_r(t) = \frac{dv_s(t)}{dt} = -2e^{-2t}, \quad t > 0.$$

At $t = 0$

$$v_s(0) = 1, v_c(0) = 1 \Longrightarrow v_r(0) = 0.$$

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**Problem 8.2:** Determine $v_{out}(t)$ for $t > 0$ for the circuit in Figure 2 if $v_{out}(0) = 0$.

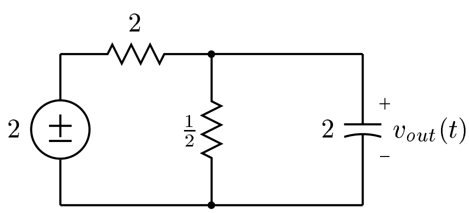


Figure 2: Circuit for Problem 8.2.

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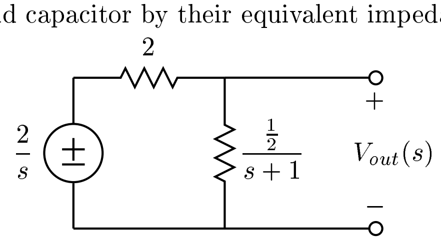
**Solution:** We begin by redrawing the circuit in the Laplace domain and replacing the parallel resistor and capacitor by their equivalent impedance.




We can now determine $V_{out}(s)$ by using a voltage divider.

$$V_{out}(s) = \frac{\frac{\frac{1}{2}}{s+1}}{2 + \frac{\frac{1}{2}}{s+1}} \cdot \frac{2}{s} = \frac{\frac{1}{2}}{s(s + \frac{5}{4})}$$

$$= \frac{A}{s} + \frac{B}{s + \frac{5}{4}}$$

$$A = \lim_{s \to 0} \frac{\frac{1}{2}}{s + \frac{5}{4}} = \frac{2}{5}$$

$$B = \lim_{s \to -\frac{5}{4}} \frac{\frac{1}{2}}{s} = -\frac{2}{5}$$

Therefore,

$$v_{out}(t) = \frac{2}{5}(1 - e^{-1.25t}), \quad t > 0$$

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**Problem 8.3:** For the circuit shown in Figure 3.

(a) Find $v_r(t)$ for $t > 0$ if $i_\ell(0) = 0$.

(b) Find $v_r(t)$ for $t > 0$ if $i_\ell(0) = 5$.

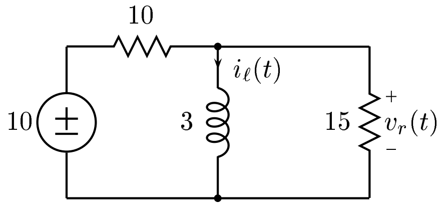


Figure 3: Circuit for Problem 8.3.

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**Solution:**

(a) At $t = 0$ the inductor looks like an open circuit.

$$v_r(0) = \frac{15}{25} \cdot 10 = 6$$

At $t = \infty$ the inductor looks like a short circuit.

$$v_r(\infty) = 0.$$

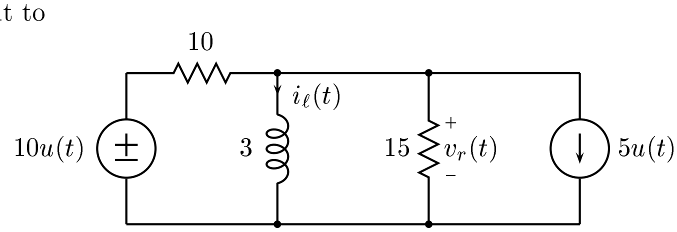
We turn off the source to determine the time constant

$$\tau = \frac{L}{R_{eq}} = \frac{3}{6} = \frac{1}{2}.$$

Therefore,

$$v_r(t) = 6e^{-2t} \quad t \geq 0.$$

(b) We can incorporate a current source to handle the initial condition. This changes the circuit to




The time constant is the same as before. The value at $t = \infty$ is also the same as before. At $t = 0$

$$v_r(0) = 6 - 30 = -24.$$

Therefore,

$$v_r(t) = -24e^{-2t} \quad t \geq 0.$$

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**Problem 8.4:** For the circuit in Figure 4, let $i_s(t) = 1$ for $t > 0$ and assume that at $t = 0$ the current through the inductor is zero and that the voltage drop across the capacitor terminals is 1 volt.

(a) Redraw the circuit in the Laplace domain.

(b) Determine $V(s)$, the Laplace transform of the resistor voltage.

(c) Determine $v(t)$ for $t > 0$.

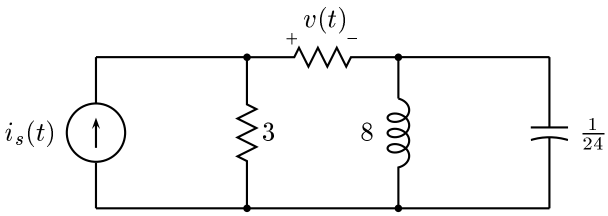


Figure 4: Circuit for Problem 8.4.

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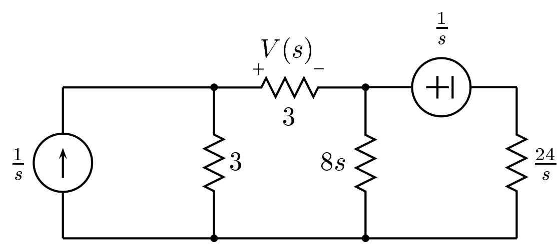
**Solution:**

(a) We replace the inductor and capacitor by impedances and insert a voltage source to realize the initial voltage on the capacitor.




(b) Let $\frac{1}{s}$, $I_\alpha(s)$, and $I_\beta(s)$ be mesh currents in the three meshes. Then, if we write KVL equations over the two rightmost meshes, we get

$$3[I_\alpha(s) - \frac{1}{s}] + 3I_\alpha(s) + 8s[I_\alpha(s) - I_\beta(s)] = 0$$
$$8s[I_\beta(s) - I_\alpha(s)] + \frac{24}{s}I_\beta(s) = -\frac{1}{s}.$$

We can rewrite these as

$$[8s+6]I_\alpha(s) - 8sI_\beta(s) = \frac{3}{s}$$
$$-8sI_\alpha(s) + [8s + \frac{24}{s}]I_\beta(s) = -\frac{1}{s}$$

If we add the two equations together we get

$$6I_\alpha(s) + \frac{24}{s}I_\beta(s) = \frac{1}{s}$$

from which

$$I_\beta(s) = -\frac{s}{4}I_\alpha(s) + \frac{1}{3s}$$

If we substitute this result into the first equation, we get

$$I_\alpha(s) = \frac{\frac{4}{3}s + \frac{3}{2}}{s(s^2+4s+3)}$$

Finally,

$$V(s) = 3I_\alpha(s) = \frac{4s + \frac{9}{2}}{s(s^2+4s+3)}$$

(c) $V(s) = \frac{\frac{3}{2}}{s} + \frac{-\frac{1}{4}}{s+1} + \frac{-\frac{5}{4}}{s+3}$. Therefore,

$$v(t) = \frac{3}{2} - \frac{1}{4}e^{-t} - \frac{5}{4}e^{-3t}, \;\; t > 0$$

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**Problem 8.5:** In the circuit in Figure 5 solve for $v(t)$ for $t > 0$. Assume that the circuit is at initial rest.

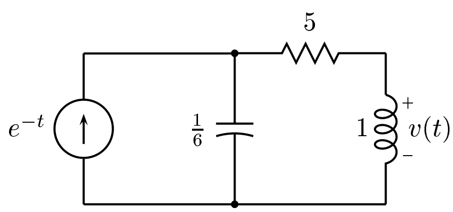


Figure 5: Circuit for Problem 8.5.

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**Solution:** Clearly

$$V(s) = sI_\ell(s)$$

and we can determine $I_\ell(s)$ by using a current divider. Therefore,

$$V(s) = s\left(\frac{\frac{6}{s}}{s+5+\frac{6}{s}}\right)\left(\frac{1}{s+1}\right) = \frac{6s}{(s+1)(s+2)(s+3)}.$$

Performing a partial fraction expansion gives

$$V(s) = \frac{-3}{s+1} + \frac{12}{s+2} + \frac{-9}{s+3}$$

and

$$v(t) = \left(-3e^{-t} + 12e^{-2t} - 9e^{-3t}\right)u(t).$$

The response is zero for $t < 0$ because the input is zero in that range.

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**Problem 8.6:** In the circuit below, $v_c(0) = 8$V and $i_\ell(0) = 4$A. Find $i_\ell(t)$ for $t > 0$.

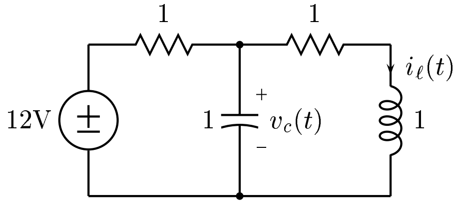


Figure 6: Circuit for Problem 8.6.

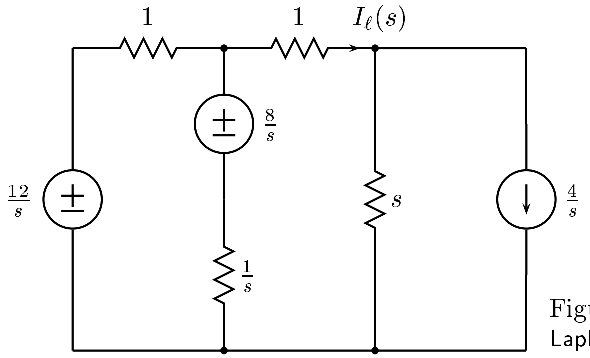


Figure 7: Circuit redrawn in the Laplace domain.

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**Solution:** In the Laplace domain the circuit can be drawn as shown in Figure 7 Let us attack this circuit using the mesh method. Let the mesh current in the left mesh be $I(s)$, in the center mesh it is the variable of interest $I_\ell(s)$, and in the right mesh we use $\frac{4}{s}$. Then the two KCL equations are

$$-\frac{12}{s} + I(s) + \frac{8}{s} + \frac{1}{s}[I(s) - I_\ell(s)] = 0$$
$$\frac{1}{s}[I_\ell(s) - I(s)] - \frac{8}{s} + I_\ell(s) + s[I_\ell(s) - \frac{4}{s}] = 0.$$

These can be simplified to

$$I(s)[s+1] - I_\ell(s) = 4$$
$$-I(s) + [s^2 + s + 1]I_\ell(s) = 4s + 8.$$

Solving for $I_\ell(s)$

$$I_\ell(s) = \frac{4(s^2 + 3s + 3)}{(s^2 + 2s + 2)s} = \frac{6}{s} + \frac{-1-j}{s+1-j} + \frac{-1+j}{s+1+j}.$$

Therefore, for $t > 0$

$$\begin{aligned} i_\ell(t) &= 6 + \sqrt{2}e^{-j3\pi/4}e^{-t}e^{jt} + \sqrt{2}e^{j3\pi/4}e^{-t}e^{-jt} \\ &= 6 + 2\sqrt{2}e^{-t}\cos(t - \frac{3\pi}{4}). \end{aligned}$$

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