GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

Course EE 6416

Multidimensional Digital Signal Processing

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Problem Set #6—Solutions

Problem 6.1: A 2-D window function is formed by taking the outer product of a 1-D window with itself.

$$w_2[n_1, n_2] = w_1[n_1]w_1[n_2]$$

The Fourier transform of the 1-D window was a main lobe width Δ and a highest side lobe height of δ. What are the main lobe and side lobe height of the 2-D window?

Solution: The Fourier transform of the 2-D window is

$$W_2(\omega_1, \omega_2) = W_1(\omega_1)W_1(\omega_2).$$

Let the height of the main lobe be A. (Note: this value was not stated in the problem. If you assumed $A = 1$, that is ok, although this would probably not be the value for a real window.) The 2-D window will have a main lobe width that is Δ in the horizontal direction and Δ in the vertical direction. The sidelobes will be highest on the two axes, where they will have a value of $A\delta$.

Problem 6.2: We wish to design a zero-phase bandpass filter whose ideal magnitude response $|H_d(\omega_1, \omega_2)|$ is given by

$$|H_d(\omega_1, \omega_2)| = \begin{cases} 1, & 0.2\pi \le \sqrt{\omega_1^2 + \omega_2^2} \le 0.4\pi \\ 0, & \text{otherwise} \end{cases}$$

The impulse response of the filter $h[n_1, n_2]$ is zero outside $-5 \leq n_1 \leq 5$, $-5 \leq n_2 \leq 5$. The filter is designed by minimizing

$$\text{error} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |H_d(\omega_1, \omega_2) - H(\omega_1, \omega_2)|^2 d\omega_1 \, d\omega_2$$

Determine $h[n_1, n_2]$.

Solution: There are two parts to this problem. The first is to recognize that the filter that minimizes the mean squared error is the design achieved with the rectangular window. That means

$$h[n_1, n_2] = \begin{cases} h_d[n_1, n_2], & -2 \leq n_1 \leq 2; -2 \leq n_2 \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

The other part of the problem is to determine $h[n_1, n_2]$. The ring-shaped bandpass filter can be expressed as the difference of two ideal circular lowpasses for which we know the impulse response. Thus,

$$h_d[n_1, n_2] = \frac{1}{\sqrt{n_1^2 + n_2^2}} \left(0.3 J_1(0.6\pi \sqrt{n_1^2 + n_2^2}) - 0. J_1(0.4\pi \sqrt{n_1^2 + n_2^2}) \right)$$

Problem 6.3: We would like to perform a (5×5)-point least-squares design of a zero-phase fan filter whose frequency response $I(\omega_1, \omega_2)$ is sketched in Figure 1. The frequency response is 1 in the shaded areas and is zero in the unshaded areas. To reduce the number of degrees of freedom in the approximation, we require that the approximating function satisfy:

1. $H(\omega_1, \omega_2)$ is purely real.
2. $H(\omega_1, \omega_2) = H(-\omega_1, \omega_2)$.
3. $H(\omega_1, \omega_2) = H(\omega_1, -\omega_2)$.

The filter $h[n_1, n_2]$ should have support over the region $-2 \leq n_1 \leq 2$, $-2 \leq n_2 \leq 2$.

(a) Of the 25 coefficients in the filter, how many are linearly independent? Express the dependent coefficients in terms of the independent ones.

(b) Express the frequency response of the approximating filter $H(\omega_1, \omega_2)$ as a linear combination of only the independent coefficients of the impulse response with an appropriate set of basis functions.

(c) If the independent filter coefficients are chosen to minimize the integral-squared error, derive an expression for their optimal values. You do not need to evaluate the integrals.

Solution:

(a) From the three symmetry conditions, we know that

$$\begin{aligned} h[n_1, n_2] &= h^*[-n_1, -n_2] \\ h[n_1, n_2] &= h[-n_1, n_2] \\ h[n_1, n_2] &= h[n_1, -n_2] \end{aligned}$$

Together these imply that the impulse response is real and four-quadrant symmetric. Thus, of the 25 coefficients, only 9 are independent.

	independent	dependent (equal)
a_1	$h[0,0]$	none
a_2	$h[1,0]$	$h[-1,0]$
a_3	$h[2,0]$	$h[-2,0]$
a_4	$h[0,1]$	$h[0,-1]$
a_5	$h[1,1]$	$h[1,-1], h[-1,1], h[-1,-1]$
a_6	$h[2,1]$	$h[2,-1], h[-2,1], h[-2,-1]$
a_7	$h[0,2]$	$h[0,-2]$
a_8	$h[1,2]$	$h[1,-2], h[-1,2], h[-1,-2]$
a_9	$h[2,2]$	$h[2,-2], h[-2,2], h[-2,-2]$

(b) The frequency response is of the form

$$H(\omega_1, \omega_2) = \sum_{i=1}^{9} a_i \phi_i(\omega_1, \omega_2)$$

where,

$$\begin{aligned} \phi_1(\omega_1, \omega_2) &= 1 \\ \phi_2(\omega_1, \omega_2) &= 2\cos(\omega_1) \\ \phi_3(\omega_1, \omega_2) &= 2\cos(2\omega_1) \\ \phi_4(\omega_1, \omega_2) &= 2\cos(\omega_2) \\ \phi_5(\omega_1, \omega_2) &= 4\cos(\omega_1)\cos(\omega_2) \end{aligned}$$

$$\begin{aligned}
\phi_6(\omega_1, \omega_2) &= 4\cos(2\omega_1)\cos(\omega_2) \\
\phi_7(\omega_1, \omega_2) &= 2\cos(2\omega_2) \\
\phi_8(\omega_1, \omega_2) &= 4\cos(\omega_1)\cos(2\omega_2) \\
\phi_9(\omega_1, \omega_2) &= 4\cos(2\omega_1)\cos(2\omega_2)
\end{aligned}$$

(c) Minimizing the integral squared error is equivalent to designing the filter using a rectangular window. For this all that is needed is to compute the appropriate samples of the ideal impulse response. Since the ideal has the same symmetries as the filter that we are trying to design, the above symmetry conditions will be incorporated automatically.

$$h[n_1, n_2] = \int \int_{\text{passband}} e^{-j(\omega_1 n_1 + \omega_2 n_2)} \, d\omega_1 \, d\omega_2$$

Problem 6.4: Although procedures have been developed for the design of transformation functions, ad hoc methods often work well since the transformation typically involves very few free parameters. Ad hoc methods may take the form of specifying the mapping function for a few key frequencies. As an example, consider a first-order transformation of the form

$$F(\omega_1, \omega_2) = A + B\cos\omega_1 + C\cos\omega_2 + D\cos\omega_1\cos\omega_2$$

to design a fan filter that approximates the ideal response shown in Figure 1.

(a) Find a reasonable set of values for A, B, C, and D. Justify your answer.

(b) Sketch the response of a 1-D prototype filter to be used with this transformation to design an approximation to Figure 1.

Solution:

(a) Assume that we will begin with a lowpass prototype filter. In order to map the center of the prototype passband ($\omega = 0$) to the center of the 2-D passband we might require that $\omega = 0$ map to $(\omega_1, \omega_2) = (0, \pi)$ and similarly that $\omega = \pi$ map to $(\omega_1, \omega_2) = (\pi, 0)$. Since $F(\omega_1, \omega_2) = \cos\omega$, these conditions say

$$A + B - C - D = 1 \qquad (1)$$

$$A - B + C - D = 1 \qquad (2)$$

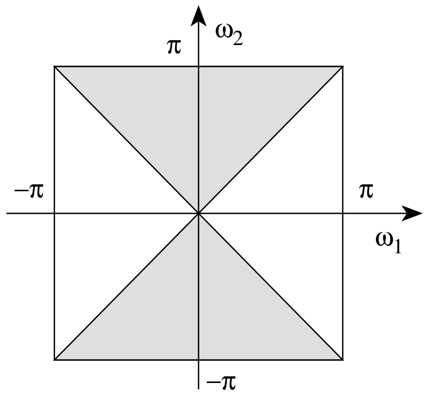

Figure 1:

We might also choose to require that the frequency response be constant on the line $\omega_1 = \omega_2$.

$$A + B\cos\omega_1 + C\cos\omega_1 + D\cos^2\omega_1 = const. \qquad (3)$$

For this to be true, we must have $D = 0$ and $C = -B$. Combining these two conditions with those from (1) and (2) gives the remaining parameters: A=0, B=0.5, and $C = -0.5$. The required transformation function is then

$$F(\omega_1, \omega_2) = 0.5(\cos\omega_1 - \cos\omega_2)\ .$$

(b) When $\omega_1 = \omega_2$, $F(\omega_1, \omega_2) = 0$, which implies that $\omega = \pi/2$. Therefore, the cutoff frequency of the 1-D lowpass prototype should be at $\pi/2$ radians. Then the cutoff frequency of the 2-D filter will fall on the indicated contours.

Problem 6.5: We would like to design a 3-D spherically symmetric, zero-phase lowpass filter using the method of transformations with a first-order transformation function of the form

$$\begin{aligned} F(\omega_1, \omega_2, \omega_3) &= A + B\cos\omega_1 + C\cos\omega_2 + D\cos\omega_3 \\ &\quad + E\cos\omega_1\cos\omega_2 + F\cos\omega_1\cos\omega_3 + G\cos\omega_2\cos\omega_3 \\ &\quad + H\cos\omega_1\cos\omega_2\cos\omega_3. \end{aligned}$$

This transformation will convert a 1-D zero-phase prototype frequency response $G(\omega)$ into the 3-D response $H(\omega_1, \omega_2, \omega_3)$ under the substitution

$$F(\omega_1, \omega_2, \omega_3) \longrightarrow \cos\omega \ .$$

Choose a set of transformation parameters A, B, ..., H so that the transformation will have the following properties:

1. $\omega = \pi$ will map to $(\omega_1, \omega_2, \omega_3) = (\pi, \pi, \pi)$.
2. $$F(\omega_1, \omega_2, \omega_3) = F(\omega_2, \omega_3, \omega_1) = F(\omega_3, \omega_1, \omega_2) = F(\omega_1, \omega_3, \omega_2) \\ = F(\omega_2, \omega_1, \omega_3) = F(\omega_3, \omega_2, \omega_1)$$
3. $F(\omega, 0, 0) = \cos\omega$. (This will guarantee that the response on each axis is the same as the prototype response.)
4. $F(\omega_1, \omega_2, \pi) = -1$

Solution: Each of the properties in the problem statement imposes one or more linear constraint on the transformation. From Property #1:

$$A - B - C - D + E + F + G - H = \cos\pi = -1 \ .$$

From Property #2:

$$B = C = D$$
$$E = F = G \ .$$

From Property #3:

$$A + B\cos\omega + C + D + E\cos\omega + F\cos\omega + G + H\cos\omega = \cos\omega$$

which implies

$$A + C + D + G = 0$$
$$B + E + F + H = 1 \ .$$

From Property #4:

$$A + B\cos\omega_1 + C\cos\omega_2 - D + E\cos\omega_1\cos\omega_2 - F\cos\omega_1 - G\cos\omega_2 - H\cos\omega_1\cos\omega_2 = -1$$

which implies

$$A - D = -1$$
$$B - F = 0$$
$$E - H = 0 \ .$$

Solving these equations gives the transformation

$$A = -0.75 \qquad B = C = D = E = F = G = H = 0.25 \ .$$