

GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

Course EE 6416

Multidimensional Digital Signal Processing

November 13, 1998

Problem Set #7—Solutions

Problem 7.1: A second-order transformation uses the frequency response of a (5×5) -point zero-phase filter as its transformation function. Assume that such a transformation function is used with a $(2N + 1)$ -point zero-phase 1-D filter to realize a 2-D filter.

- (a) How many multiplications need to be performed to compute one sample of the output sequence?
- (b) How many multiplications need to be performed per output sample if this same frequency response is implemented using direct convolution?

Solution: Like a filter designed using a first-order transformation, a filter designed using a second-order transformation can be implemented using the special implementation based on the Chebyshev recursion. All that changes are the small filters that perform a convolution with the transformation filter.

- (a) Each of the 5×5 zero-phase filters can be implemented using 13 multiplications and 24 additions. There will be N of these, plus $N + 1$ coefficients from the prototype. Therefore,

$$\# \text{ of multiplications} = 13N + N + 1 = 14N + 1 \text{ per output.}$$

$$\# \text{ of additions} = 24N + N = 25N \text{ per output.}$$

- (b) The impulse response is $(4N + 1) \times (4N + 1)$ and is zero-phase. Thus, the number of multiplications per output in a direct implementation is $8N^2 + 4N + 1$.
-

Problem 7.2: Consider a recursively computable digital filter with the output mask shown in Figure 1. This output mask corresponds to the difference equation

$$y[n_1, n_2] + ay[n_1 - 1, n_2 - 1] + by[n_1, n_2 - 1] + cy[n_1 + 1, n_2 - 1] = x[n_1, n_2].$$

- (a) Determine the region of support of the impulse response of the filter if it is to behave like a linear, shift-invariant system.
- (b) Find a sufficient set of initial conditions that will guarantee that the system is linear, shift-invariant, and recursively computable.

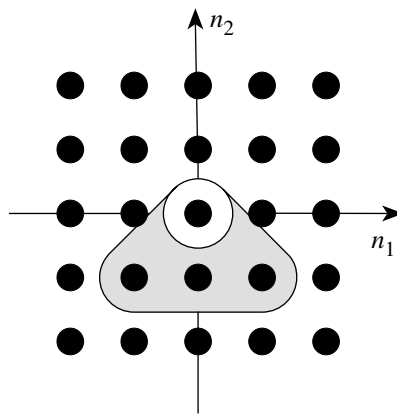


Figure 1:

Solution: The support of the filter is shown in Figure 2 as solid dots. A sufficient set of initial conditions for producing a linear, shift-invariant system borders the support of the output sequence formed as the convolution of the impulse response with the input. Since, for this problem, the input sequence is an impulse, the initial conditions merely frame the support of the filter and are shown as open dots.

Problem 7.3: Repeat the previous problem for the output mask shown in Figure P4.3 on page 209 of the notes.

Solution: The support of the filter is shown in Figure 3 as solid dots. A sufficient set of initial conditions for producing a linear, shift-invariant system borders the support of the output sequence formed

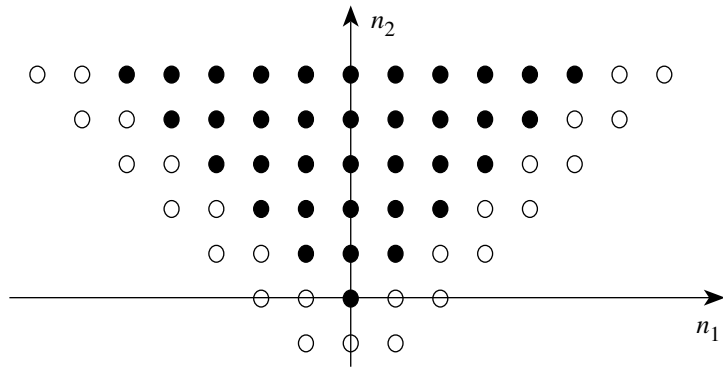


Figure 2:

as the convolution of the impulse response with the input. Since, for this problem, the input sequence is an impulse, the initial conditions merely frame the support of the filter and are shown as open dots.

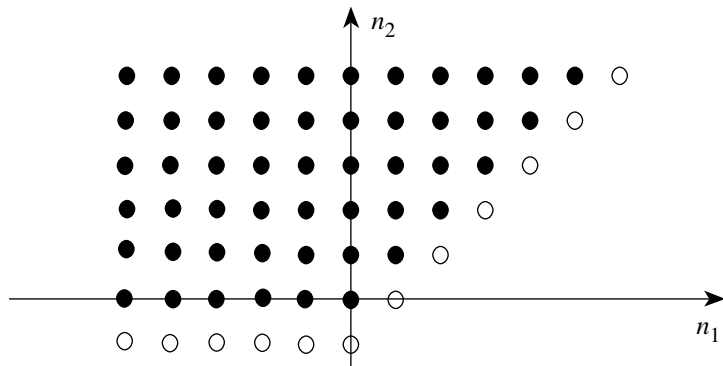


Figure 3:

Problem 7.4: Find the z -transform of the following 2-D arrays and give the associated regions of convergence.

- (a) $\delta[n_1, n_2]$.
- (b) $a^{(n_1+n_2)}u[n_1, n_2]$.
- (c)

$$x[n_1, n_2] = \begin{cases} \delta[n_1], & n_2 = 0, 1, \dots, N-1 \\ 0, & \text{otherwise} . \end{cases}$$

Solution:

(a)

$$X_a(z_1, z_2) = \sum_{n_2} \sum_{n_2} \delta[n_1, n_2] z_1^{-n_1} z_2^{-n_2} = 1, \quad \forall z_1, z_2$$

(b)

$$\begin{aligned} X_b(z_1, z_2) &= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} a^{n_1+n_2} u[n_1, n_2] z_1^{-n_1} z_2^{-n_2} \\ &= \sum_{n_1=-\infty}^{\infty} a^{n_1} z_1^{-n_1} \cdot \sum_{n_2=-\infty}^{\infty} a^{n_2} z_2^{-n_2} \\ &= \frac{1}{1 - az_1^{-1}} \cdot \frac{1}{1 - az_2^{-1}}, \quad |z_1| > a, |z_2| > a \end{aligned}$$

(c)

$$\begin{aligned} X_c(z_1, z_2) &= \sum_{n_2=0}^{N-1} \sum_{n_1=-\infty}^{\infty} \delta[n_1] z_1^{-n_1} z_2^{-n_2} \\ &= \sum_{n_2=0}^{N-1} z_2^{-n_2} = \frac{1 - z_2^{-N}}{1 - z_2^{-1}}, \quad \forall z_1, z_2 \end{aligned}$$

Problem 7.5: In Chapter 4 we stated a number of properties of the z -transform without proof. These properties are derived in this problem.

(a) Derive the following properties of the z -transform given that $X_z(z_1, z_2)$ is the z -transform of $x[n_1, n_2]$.

(a) $x^*[n_1, n_2] \longleftrightarrow X_z^*(z_1^*, z_2^*)$

(b) $\Re\{x[n_1, n_2]\} \longleftrightarrow \frac{1}{2} \{X_z(z_1, z_2) + X_z^*(z_1^*, z_2^*)\}$

(c) $\Im\{x[n_1, n_2]\} \longleftrightarrow \frac{1}{2j} \{X_z(z_1, z_2) - X_z^*(z_1^*, z_2^*)\}$

(d) $x[-n_1, n_2] \longleftrightarrow X_z(z_1^{-1}, z_2)$

(e) $x[n_1, -n_2] \longleftrightarrow X_z(z_1, z_2^{-1})$

(f) $x[-n_1, -n_2] \longleftrightarrow X_z(z_1^{-1}, z_2^{-1})$

(g) $x[n_2, n_1] \longleftrightarrow X_z(z_2, z_1)$

(b) Derive Parseval's Theorem

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x[n_1, n_2]x^*[n_1, n_2] = \frac{1}{(2\pi j)^2} \oint_{C_2} \oint_{C_1} X_z(z_1, z_2)Y_z\left(\frac{1}{z_1^*}, \frac{1}{z_2^*}\right)z_1^{-1}z_2^{-1} dz_1 dz_2.$$

Solution:

(a) (1)

$$\begin{aligned} X_z(z_1, z_2) &= \sum_{n_1} \sum_{n_2} x[n_1, n_2]z_1^{-n_1}z_2^{-n_2} \\ X_z^*(z_1, z_2) &= \sum_{n_1} \sum_{n_2} x^*[n_1, n_2](z_1^*)^{-n_1}(z_2^*)^{-n_2} \\ X_z^*(z_1^*, z_2^*) &= \sum_{n_1} \sum_{n_2} x^*[n_1, n_2]z_1^{-n_1}z_2^{-n_2} \end{aligned}$$

Therefore

$$x^*[n_1, n_2] \longleftrightarrow X_z^*(z_1^*, z_2^*)$$

(2) Since

$$\Re\{x[n_1, n_2]\} = \frac{1}{2}(x[n_1, n_2] + x^*[n_1, n_2])$$

we can apply the linearity property and the result in part (1) to establish the result

$$\Re\{x[n_1, n_2]\} \longleftrightarrow \frac{1}{2}\{X_z(z_1, z_2) + X_z^*(z_1^*, z_2^*)\}$$

(3) Here you can use the same approach that you used in part (b).

(4)

$$\begin{aligned} \sum_{n_1} \sum_{n_2} x[-n_1, n_2]z_1^{-n_1}z_2^{-n_2} &= \sum_{-n_1} \sum_{n_2} x[n_1, n_2]z_1^{n_1}z_2^{-n_2} \\ &= X_z(z_1^{-1}, z_2) \end{aligned}$$

(5) This one is nearly identical to (4) above.

(6) Apply (4) and (5) sequentially.

(7)

$$\begin{aligned} \sum_{n_1} \sum_{n_2} x[n_2, n_1]z_1^{-n_1}z_2^{-n_2} &= \sum_{m_1} \sum_{m_2} x[m_1, m_2]z_2^{-m_1}z_1^{-m_2} \\ &= X_z(z_2, z_1) \end{aligned}$$

We have made the substitution of variables $m_1 = n_2$ and $m_2 = n_1$.

(b)

$$\begin{aligned} & \sum_{n_1} \sum_{n_2} x[n_1, n_2] y^*[n_1, n_2] \\ &= \sum_{n_1} \sum_{n_2} y^*[n_1, n_2] \frac{1}{(2\pi j)^2} \oint_{C_2} \oint_{C_1} X_z(z_1, z_2) z_1^{n_1-1} z_2^{n_2-1} dz_1 dz_2 \\ &= \frac{1}{(2\pi j)^2} \oint_{C_2} \oint_{C_1} X_z(z_1, z_2) z_1^{-1} z_2^{-1} \sum_{n_1} \sum_{n_2} y^*[n_1, n_2] z_1^{n_1} z_2^{n_2} dz_1 dz_2 \\ &= \frac{1}{(2\pi j)^2} \oint_{C_2} \oint_{C_1} X_z(z_1, z_2) Y_z\left(\frac{1}{z_1^*}, \frac{1}{z_2^*}\right) z_1^{-1} z_2^{-1} dz_1 dz_2 \end{aligned}$$
