GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

Course EE 6416

Multidimensional Digital Signal Processing

October 16, 1998

Problem Set #3–Solutions

Problem 3.1: A two-dimensional bandlimited signal has the elliptical support

$$H(\Omega_1, \Omega_2) = 0 \qquad \text{if } \frac{\Omega_1^2}{a^2} + \frac{\Omega_2^2}{b^2} \geq 1 \ .$$

(a) Determine a sampling matrix $\boldsymbol{V}$ that will allow $h(t_1, t_2)$ to be sampled at a minimum sampling rate without aliasing.

(b) What is the minimum sampling density (in samples per unit area)?

Solution:

(a) If $a = b$, the optimal sampling raster is hexagonal with a sampling matrix

$$\boldsymbol{V} = \begin{bmatrix} \frac{\pi}{a\sqrt{3}} & \frac{\pi}{a\sqrt{3}} \\ \frac{\pi}{a} & -\frac{\pi}{a} \end{bmatrix} .$$

Scaling the vertical bandwidth by b/a (An ellipse is simply a circle on which different scaling has been applied to the horizontal and vertical variables.) scales the sampling period in the vertical direction by a/b. Therefore,

$$\boldsymbol{V} = \begin{bmatrix} \frac{\pi}{a\sqrt{3}} & \frac{\pi}{a\sqrt{3}} \\ \frac{\pi}{b} & -\frac{\pi}{b} \end{bmatrix} .$$

(b) $|\det(\boldsymbol{V})| = \frac{2\pi^2}{ab\sqrt{3}} \Longrightarrow$ sampling density $= \frac{ab\sqrt{3}}{2\pi^2}$.

Problem 3.2: Analog bandlimited waveforms have Fourier transforms with the regions of support indicated in Figure 1. For each determine the minimum sampling density (in samples per square meter) that will permit an exact reconstruction of the analog waveform. For each case sketch the optimal sampling raster.

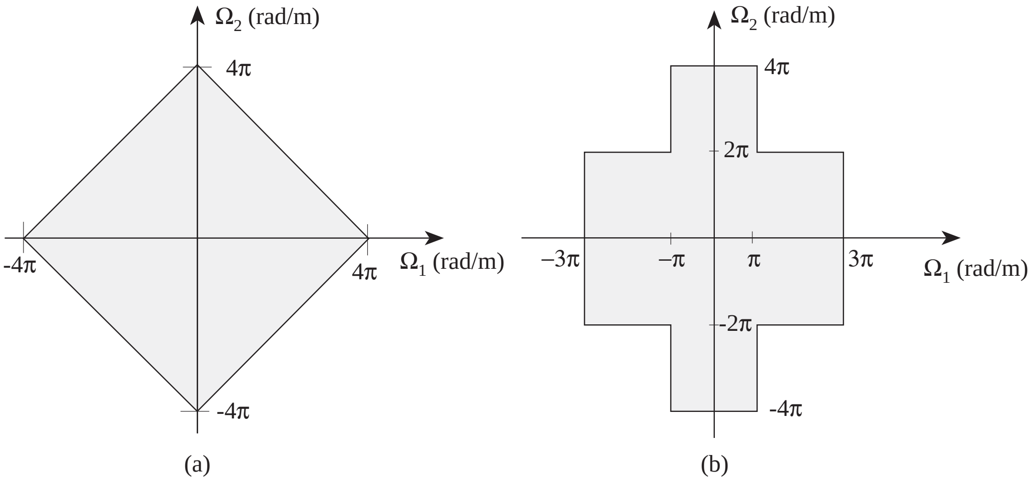

Figure 1:

Solution: Each of these waveforms can be optimally sampled. The two aliased spectra are shown in Figure 2. For both tilings of the frequency plane a valid $\boldsymbol{U}$-matrix is:

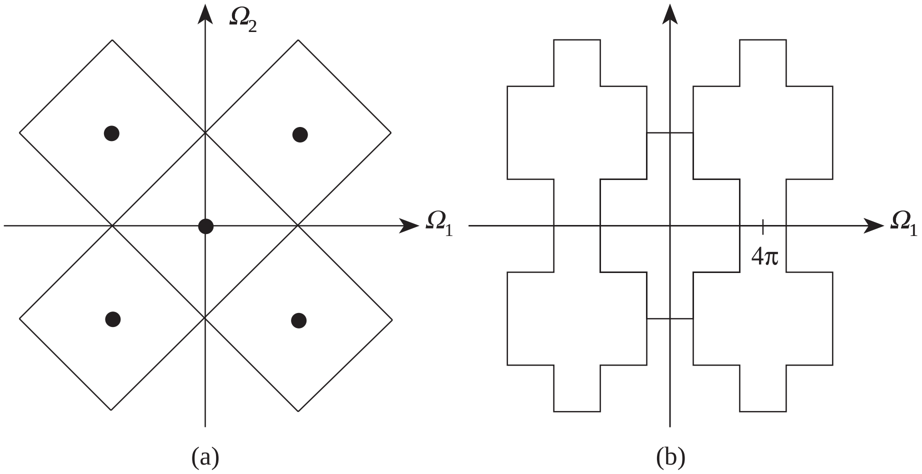

Figure 2:

$$\boldsymbol{U} = \begin{bmatrix} 4\pi & 4\pi \\ 4\pi & -4\pi \end{bmatrix} .$$

for which the sampling density is

$$\frac{1}{4\pi^2} |\det(\boldsymbol{U})| = 8 \text{ samples/m}^2 .$$

$$\boldsymbol{V} = 2\pi \boldsymbol{U}^{-t} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix} .$$

The sampling lattice is shown in Figure 3.

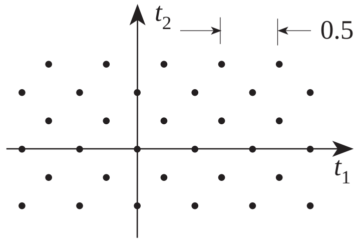

Figure 3:

Problem 3.3: A signal $x(t_1, t_2)$, with the bandlimited spectrum shown in Figure 4, is sampled at the sampling points sketched in Figure 5.

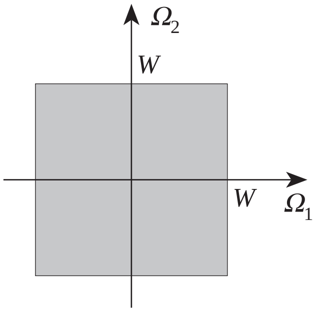

Figure 4:

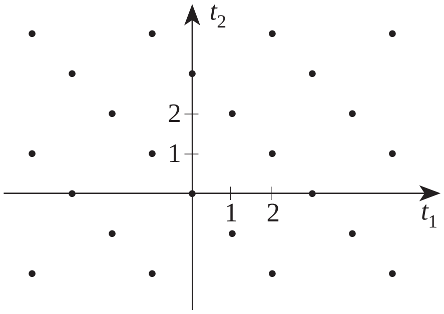

Figure 5:

(a) Determine a sampling matrix, $\boldsymbol{V}$.

(b) Determine the corresponding aliasing matrix, $\boldsymbol{U}$.

(c) Sketch the spectrum of the sampled signal with respect to the continuous frequencies (Ω_1, Ω_2).

(d) Determine the largest value for the bandwidth, W, if there is to be no aliasing.

Solution:

(a) The sampling matrix is not unique. Any matrix will do provided that the columns of that matrix form a basis for the lattice. This means that the basis vectors must be linearly independent, and that the parallelogram formed with the two vectors as sides must not enclose any other lattice samples. One choice is

$$\boldsymbol{V} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}.$$

For any correct choice, however, $|\det \boldsymbol{V}| = 3$.

(b) The sampling matrix and the aliasing matrix must satisfy the relationship

$$\boldsymbol{U}^t \boldsymbol{V} = 2\pi \boldsymbol{I}.$$

Therefore, the matrix $\boldsymbol{U}$ that goes with the above sampling matrix is

$$\boldsymbol{U} = \begin{bmatrix} 4\pi/3 & 2\pi/3 \\ -2\pi/3 & 2\pi/3 \end{bmatrix}.$$

(c) The spectrum of the sampled signal is shown in Figure 6.

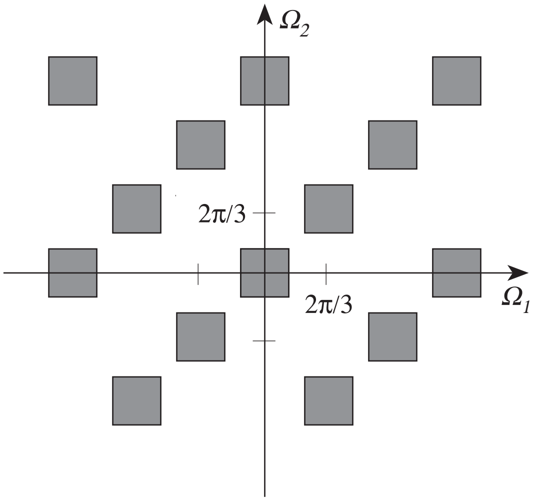

Figure 6:

(d) Aliasing will occur if $W \geq \pi/3$.

Problem 3.4: A 3-D signal $x[n_1, n_2, n_3]$ is isotropically bandlimited with a spectrum that satisfies

$$X(\Omega_1, \Omega_2, \Omega_3) = 0, \qquad \text{if } \Omega_1^2 + \Omega_2^2 + \Omega_3^2 \geq (10\pi)^2 \ .$$

(a) What is the minimum sampling density required (in samples per unit volume) if the signal is sampled rectangularly?

(b) The optimal sampling raster places the replicated copies of the bandlimited spectrum at the vertices of a dodecahedron (face-centered cubic lattice). The polar (aliasing) lattice is defined by the matrix

$$\boldsymbol{U} = \begin{bmatrix} \frac{20\pi}{\sqrt{2}} & -\frac{20\pi}{\sqrt{2}} & \frac{20\pi}{\sqrt{2}} \\ \frac{20\pi}{\sqrt{2}} & 0 & -\frac{20\pi}{\sqrt{2}} \\ 0 & \frac{20\pi}{\sqrt{2}} & 0 \end{bmatrix} .$$

Determine the ratio of the minimum sampling rate required with this lattice to that of the optimal rectangular lattice if no aliasing is to occur.

Solution:

(a) A $\boldsymbol{U}$-matrix that will work for the rectangular sampling case is

$$\boldsymbol{U} = \begin{bmatrix} 20\pi & 0 & 0 \\ 0 & 20\pi & 0 \\ 0 & 0 & 20\pi \end{bmatrix} .$$

For this lattice the sampling density is

$$\frac{1}{|\det \boldsymbol{V}|} = \frac{|\det \boldsymbol{U}|}{(2\pi)^3} = \frac{8000\pi^3}{8\pi^3} = 1000 \text{ samples/vol.}$$

(b)

$$|\det \boldsymbol{U}| = \left| \frac{8000\pi^3}{(\sqrt{2})^3} + \frac{8000\pi^3}{(\sqrt{2})^3} \right| = \frac{8000\pi^3}{\sqrt{2}}$$

$$\frac{1}{|\det \boldsymbol{V}|} = \frac{8000\pi^3}{8\pi^3\sqrt{2}} = \frac{1000}{\sqrt{2}} \text{ samples/vol.} = 705 \text{ samples/vol.}$$

$$\text{ratio} = \frac{1}{\sqrt{2}} = 0.705$$

Problem 3.5: Suppose that $\tilde{x}[n_1, n_2]$ is a rectangularly periodic sequence with horizontal period N_1 and vertical period N_2. The sequence $\tilde{x}_1[n] = \tilde{x}[n, n]$ is then a periodic one-dimensional sequence.

(a) Show that $\tilde{x}[n]$ is a periodic sequence with period $N_1 N_2$. Show that if N_1 and N_2 have any common integral factors then $\tilde{x}[n]$ will also have a smaller period.

(b) Assuming that N_1 and N_2 have no common factors, show that the samples of the DFS coefficients $\tilde{X}_1[k]$ are equal to selected values of $\tilde{X}[k_1, k_2]$ and determine the mapping between k and $[k_1, k_2]$.

If the row-column algorithm is used to evaluate the 2-D DFS coefficients, $\tilde{X}[k_1, k_2]$ this is an efficient algorithm for computing the one-dimensional DFS $\tilde{X}_1[k]$ known as the *prime factor algorithm.*

Solution:

(a)

$$\begin{aligned}
\tilde{x}_1[n + N_1N_2] &= \tilde{x}_2[n + N_1N_2, n + N_1N_2] \\
&= \tilde{x}_2[n, n + N_1N_2] \text{ (by horizontal periodicity)} \\
&= \tilde{x}_2[n, n] \text{ (by vertical periodicity)} \\
&= \tilde{x}_1[n].
\end{aligned}$$

In the more general case the period is $N = N_1N_2/\gcd(N_1, N_2)$, where $\gcd(.,.)$ is the greatest common divisor. Since N is a multiple of both N_1 and N_2, the proof is the same as the one above.

(b)

$$\begin{aligned}
\tilde{X}_1[k] &= \sum_{n=0}^{N_1N_2-1} \tilde{x}_2[n, n] W_{N_1N_2}^{nk} \\
&= \sum_{n=0}^{N_1N_2-1} \frac{1}{N_1N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \tilde{X}_2[k_1, k_2] W_{N_1}^{-nk_1} W_{N_2}^{-nk_2} W_{N_1N_2}^{nk} \\
&= \frac{1}{N_1N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \tilde{X}_2[k_1, k_2] \sum_{n=0}^{N_1N_2-1} \left[W_{N_1}^{-k_1} W_{N_2}^{-k_2} W_{N_1N_2}^{k} \right]^n .
\end{aligned}$$

The innermost sum is zero unless $k = N_1k_2 + N_2k_1$. Therefore,

$$\tilde{X}_1[k] = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \tilde{X}_2[k_1, k_2] \delta[k - N_1k_2 - N_2k_1] .$$

Since N_1 and N_2 are relatively prime, each value of $[k_1, k_2]$ over the range of summation contributes to only one value of k. The samples in $\tilde{X}_1[k]$ are simply the samples of $\tilde{X}_2[k_1, k_2]$, scrambled.