

GEORGIA INSTITUTE OF TECHNOLOGY  
School of Electrical and Computer Engineering

Course ECE 2040  
Circuit Analysis

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Problem Set #7–Solutions

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**Problem 7.1:** Find the inverse Laplace transform of

$$X(s) = \frac{s + 3}{s^3 + 3s^2 + 6s + 4}.$$

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**Solution:**

$$\begin{aligned} X(s) &= \frac{s + 3}{s^3 + 3s^2 + 6s + 4} = \frac{s + 3}{(s + 1)(s^2 + 2s + 4)} \\ &= \frac{s + 3}{(s + 1)(s + 1 - j\sqrt{3})(s + 1 + j\sqrt{3})} \\ &= \frac{A}{s + 1} + \frac{B}{s + 1 - j\sqrt{3}} + \frac{B^*}{s + 1 + j\sqrt{3}}. \end{aligned}$$

$$A = \lim_{s \rightarrow -1} \frac{s + 3}{s^2 + 2s + 4} = \frac{2}{3}$$

$$B = \lim_{s \rightarrow (-1 + j\sqrt{3})} \frac{s + 3}{(s + 1)(s + 1 + j\sqrt{3})} = -\frac{1}{3} - j\frac{1}{2\sqrt{3}}$$

Therefore,

$$\begin{aligned} x(t) &= \frac{2}{3}e^{-t} + \left(-\frac{1}{3} - j\frac{1}{2\sqrt{3}}\right)e^{-t}e^{j\sqrt{3}t} + \left(-\frac{1}{3} + j\frac{1}{2\sqrt{3}}\right)e^{-t}e^{-j\sqrt{3}t} \\ &= \frac{2}{3}e^{-t} - \frac{1}{3}e^{-t}(e^{j\sqrt{3}t} + e^{-j\sqrt{3}t}) + \frac{1}{j2\sqrt{3}}e^{-t}(e^{j\sqrt{3}t} - e^{-j\sqrt{3}t}) \\ &= \frac{2}{3}e^{-t} - \frac{2}{3}e^{-t}\cos\sqrt{3}t + \frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t, \quad t > 0 \end{aligned}$$

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**Problem 7.2:** Find the inverse Laplace transform of

$$X(s) = \frac{5s - 1}{s^3 - 3s - 2}.$$

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**Solution:**

$$\begin{aligned} X(s) &= \frac{5s-1}{s^3-3s+2} = \frac{5s-1}{(s+1)^2(s-2)} \\ &= \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{(s+1)^2} \end{aligned}$$

$$A = \lim_{s \rightarrow 2} \frac{5s-1}{(s+1)^2} = 1$$

$$B = \lim_{s \rightarrow -1} \frac{d}{ds} \left( \frac{5s-1}{s-2} \right) = \lim_{s \rightarrow -1} \frac{(s-2)5 - (5s-1)}{(s-2)^2} = -1$$

$$C = \lim_{s \rightarrow -1} \frac{5s-1}{s-2} = 2$$

Therefore,

$$x(t) = e^{-2t} - e^{-t} + 2te^{-t} = e^{-2t} + (2t-1)e^{-t}, \quad t > 0.$$

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**Problem 7.3:** Find the signal  $x(t)$  for  $t \geq 0$  if its Laplace transform is

$$X(s) = \frac{s^3}{(s+1)([s+1]^2+4)}.$$

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**Solution:**

$$X(s) = \frac{s^3}{(s+1)(s^2+2s+5)} = 1 + \frac{A}{s+1} + \frac{B}{s+1-j2} + \frac{B^*}{s+1+j2}$$

$$A = \lim_{s \rightarrow -1} \frac{s^3}{s^2+2s+5} = -\frac{1}{4}$$

$$B = \lim_{s \rightarrow -1+j2} \frac{s^3}{(s+1)(s+1+j2)} = -\frac{11}{8} - \frac{1/2}{j2}$$

Therefore,

$$\begin{aligned} x(t) &= \delta(t) - \frac{1}{4}e^{-t} + \left(-\frac{11}{8} + \frac{1/2}{j2}\right)e^{-t}e^{j2t} + \left(-\frac{11}{8} - \frac{1/2}{j2}\right)e^{-t}e^{-j2t} \\ &= \delta(t) - e^{-t} \left( \frac{1}{4} + \frac{11}{4} \cos 2t - \frac{1}{2} \sin 2t \right), \quad t \geq 0 \end{aligned}$$

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**Problem 7.4:** Determine the differential equation whose solution is the voltage drop  $v(t)$  in the circuit in Figure 1. Also determine the value of  $v(0)$ .

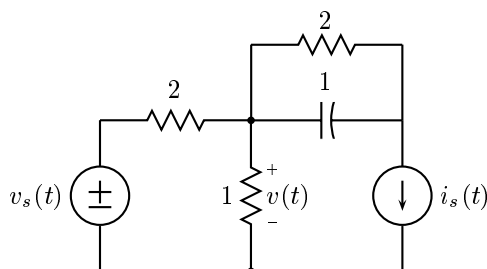


Figure 1: A first-order circuit with two sources for Problem 6.2.

**Solution:** Let  $v_c(t)$  be the voltage drop across the capacitor and  $v_r(t)$  be the voltage drop across the leftmost  $2\Omega$  resistor. Then we can write one KVL and two KCL equations in the three voltage variables,  $v_c(t)$ ,  $v_r(t)$ , and  $v(t)$ .

$$\text{KVL: } v_r(t) + v(t) = v_s(t)$$

$$\text{KCL1: } -\frac{1}{2}v_r(t) + i_c(t) + \frac{1}{2}v_c(t) = 0$$

$$\text{KCL2: } i_c(t) + \frac{1}{2}v_c(t) = i_s(t)$$

where  $i_c(t) = \frac{dv_c}{dt}$ . Substituting for  $i_c(t) + \frac{1}{2}v_c(t)$  from the third equation into the second gives

$$\begin{aligned} v_r(t) + v(t) &= v_s(t) \\ -\frac{1}{2}v_r(t) &= -i_s(t) \implies v_r(t) = 2i_s(t) \end{aligned}$$

Finally,

$$v(t) = v_s(t) - 2i_s(t),$$

which contains no derivative terms.

**Problem 7.5:** Determine the differential equation and initial condition that relates  $v_{out}(t)$  to  $v_{in}(t)$  in the circuit in Figure 2. Assume that the current flowing through the inductor is zero at  $t = 0$ .

**Solution:** Writing KCL at the inverting input of the operational amplifier

$$i_\ell(t) + \frac{1}{10}v_{out}(t) = 0,$$

where  $i_\ell(t)$  is the current flowing through the inductor. A KVL equation around the input mesh gives

$$\begin{aligned} v_{in}(t) &= 1 \cdot i_\ell(t) + v_\ell(t) \\ &= i_\ell(t) + \frac{1}{2} \frac{di_\ell(t)}{dt} \end{aligned}$$

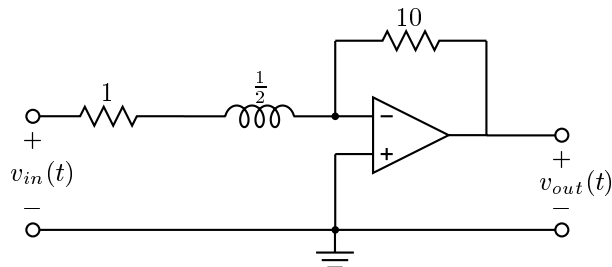


Figure 2: A first-order circuit at initial rest for Problem 6.3.

Substituting for  $i_\ell(t)$  gives

$$v_{in}(t) = -\frac{1}{10}v_{out}(t) - \frac{1}{20}\frac{dv_{out}(t)}{dt}.$$

With minor rewriting this becomes

$$\frac{1}{2}\frac{dv_{out}(t)}{dt} + v_{out}(t) = -10v_{in}(t).$$

At  $t = 0$ ,  $i_\ell(t) = 0$ , which implies  $v_{out}(t) = 0$  since no current will flow through the feedback resistor.

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**Problem 7.6:** Determine the solution of each of the following differential equations for  $t > 0$ .

- (a)  $\frac{dy(t)}{dt} + 2y(t) = 3, \quad y(0) = 0$
- (b)  $2\frac{dy(t)}{dt} + y(t) = e^{-t}, \quad y(0) = 0$
- (c)  $\frac{dy(t)}{dt} + \frac{1}{5}y(t) = e^{-t}e^{jt}, \quad y(0) = 1$

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**Solution:**

- (a) For the forced solution assume  $y(t) = Y$ . Substituting into the differential equation gives

$$2Y = 3 \implies Y = 1.5.$$

The homogeneous equation is

$$\frac{dy_h(t)}{dt} + 2y_h(t) = 0.$$

Therefore, the homogeneous solution is

$$y_h(t) = Ae^{-2t}$$

and

$$y(t) = 1.5 + Ae^{-2t}.$$

For  $y(0) = 0$ , we select  $A = -1.5$ . Therefore, the total solution is

$$y(t) = 1.5(1 - e^{-2t}), \quad t \geq 0.$$

- (b) For the forced solution try  $y_f(t) = Ye^{-t}$ . Substituting into the differential equation gives

$$-2Ye^{-t} + Ye^{-t} = e^{-t}.$$

For this equation to hold, we must have  $Y = -1$ . The homogeneous equation is

$$2\frac{dy_h(t)}{dt} + y_h(t) = 0,$$

and its solution is

$$y_h(t) = Ae^{-t/2}.$$

Therefore,

$$y(t) = -e^{-t} + Ae^{-t/2}.$$

For  $y(0) = 0$ , we select  $A = 1$ . Then

$$y(t) = e^{-t/2} - e^{-t}, \quad t \geq 0.$$

- (c) For the forced solution we try  $y_f(t) = Ye^{-(1+j)t}$ . Substituting into the differential equation

$$(-1 + j)Ye^{-(1+j)t} + \frac{1}{5}Ye^{-(1+j)t} = e^{-(1+j)t}.$$

Solving for  $Y$  gives

$$Y = \frac{1}{-\frac{4}{5} + j} = \frac{-20 - j5}{41}.$$

The homogeneous equation is

$$\frac{dy_h(t)}{dt} + \frac{1}{5}y_h(t) = 0$$

whose solution is

$$y_h(t) = Ae^{-t/5}.$$

This gives for the form of the total solution

$$y(t) = \frac{-20 - j5}{41}e^{-(1+j)t} + Ae^{-t/5}.$$

Using the initial condition, the value of  $A$  is seen to be

$$A = 1 - \frac{20}{41} - \frac{j5}{41} = \frac{21 - j5}{41}.$$

Therefore,

$$y(t) = \frac{1}{41} \left[ (-20 - j5)e^{-(1+j)t} + (21 - j5)e^{-t/5} \right], \quad t \geq 0.$$