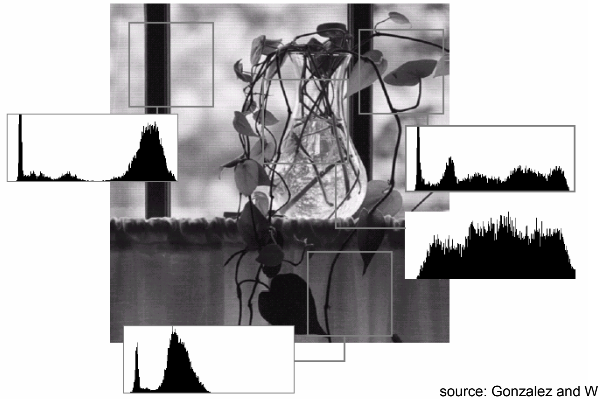
ECE6258 Lecture 23

Multiresolution Processing

Motivation

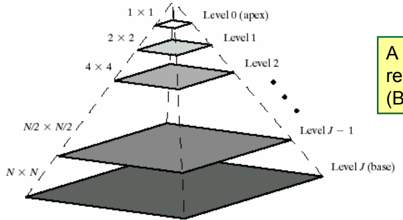
- Fourier transforms
  - Basis functions are sinusoids of infinite duration
- Wavelet transforms
  - Basis functions are signals (wavelets) of varying frequency and *limited duration*.
  - Formalized in 1987
  - Allows for a representation and analysis of signals at more than one resolution.

Local statistical variation in images

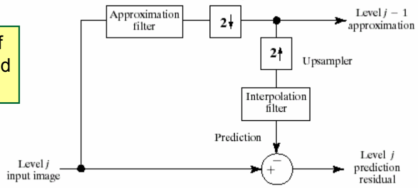


source: Gonzalez and Woods

Image pyramids

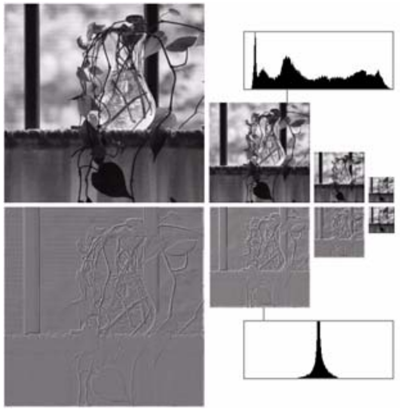


A collection of decreasing resolution images (Burt and Adelson)



Total number of pixels in pyramid = 4/3 N^2

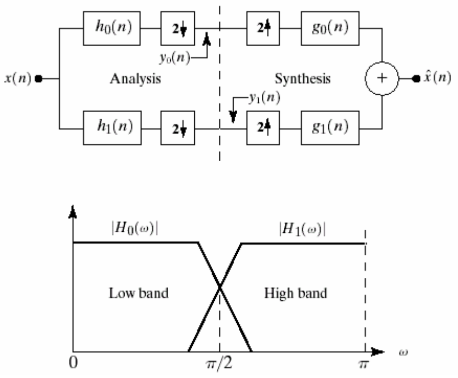
Gaussian and Laplacian pyramids



Gaussian pyramid

Laplacian pyramid

Subband Decompositions



- A (1-D) signal is decomposed into a lowpass and a highpass component.
- Each is *critically sampled*, so no extra samples are involved.

Upsampling and Downsampling

$$x[n] \longrightarrow \boxed{\downarrow 2} \longrightarrow x_{down}[n]$$

$$x[n] \longrightarrow \boxed{\uparrow 2} \longrightarrow x_{up}[n]$$

$$x_{down} = x[2n]$$

$$X_{down}(z) = \frac{1}{2}\left(X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})\right)$$

$$X_{down}(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega/2}) + X(e^{j(\omega/2+\pi)})\right)$$

DOWNSAMPLER

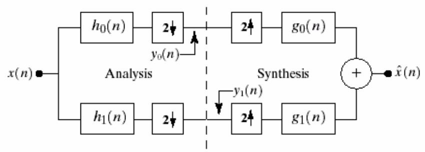
$$x_{up} = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$X_{up}(z) = X(z^2)$$

$$X_{up}(e^{j\omega}) = X(e^{j2\omega})$$

UPSAMPLER

Analysis of System



$$Y_0(z) = \frac{1}{2}\left[X(z^{\frac{1}{2}})H_0(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})H_0(-z^{\frac{1}{2}})\right]$$

$$Y_1(z) = \frac{1}{2}\left[X(z^{\frac{1}{2}})H_1(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})H_1(-z^{\frac{1}{2}})\right]$$

$$\hat{X}(z) = \frac{1}{2}G_0(z)\left[H_0(z)X(z) + H_0(-z)X(-z)\right] + \frac{1}{2}G_1(z)\left[H_1(z)X(z) + H_1(-z)X(-z)\right]$$

Aliasing terms

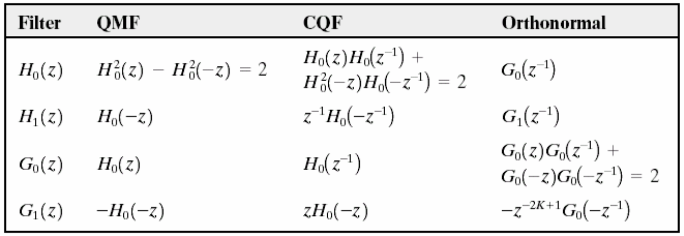
System Analysis (cont'd)

- In order to have $\hat{x}[n] = x[n]$ (perfect reconstruction)

$$\begin{aligned} H_0(-z)G_0(z) + H_1(-z)G_1(z) &= 0 \\ H_0(z)G_0(z) + H_1(z)G_1(z) &= 2 \end{aligned}$$

- The first equation guarantees aliasing cancellation.
- The second corrects the amplitude distortion.

Perfect reconstruction filter families

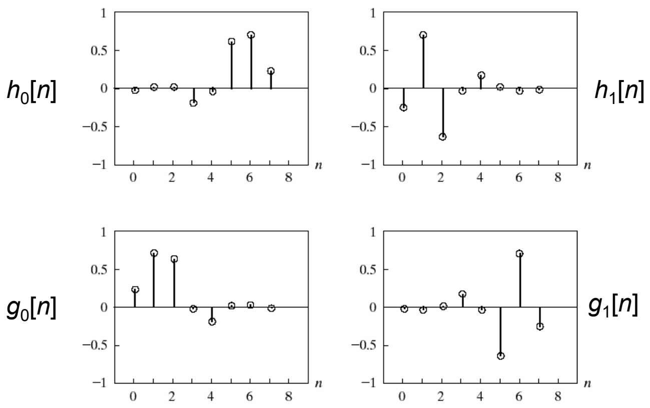


Filter	QMF	CQF	Orthonormal
$H_0(z)$	$H_0^2(z) - H_0^2(-z) = 2$	$H_0(z)H_0(z^{-1}) +$ $H_0^2(-z)H_0(-z^{-1}) = 2$	$G_0(z^{-1})$
$H_1(z)$	$H_0(-z)$	$z^{-1}H_0(-z^{-1})$	$G_1(z^{-1})$
$G_0(z)$	$H_0(z)$	$H_0(z^{-1})$	$G_0(z)G_0(z^{-1}) +$ $G_0(-z)G_0(-z^{-1}) = 2$
$G_1(z)$	$-H_0(-z)$	$zH_0(-z)$	$-z^{-2K+1}G_0(-z^{-1})$

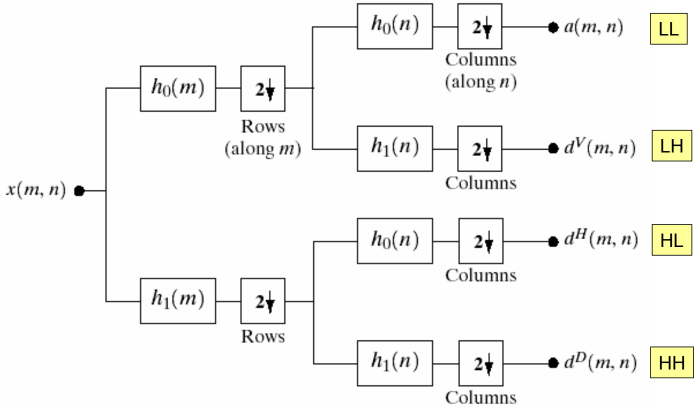
$$\begin{aligned} g_1[n] &= (-1)^n g_0[2K-1-n] \\ h_i[n] &= g_i[2K-1-n], \quad i = \{0,1\} \end{aligned}$$

source: Gonzalez and Woods

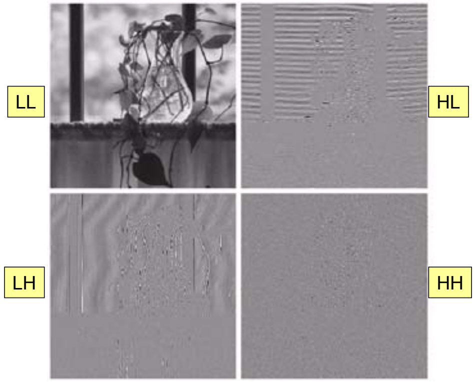
8-tap Daubechies orthonormal filters



2-D subband decomposition (one stage)



Result of four-band split



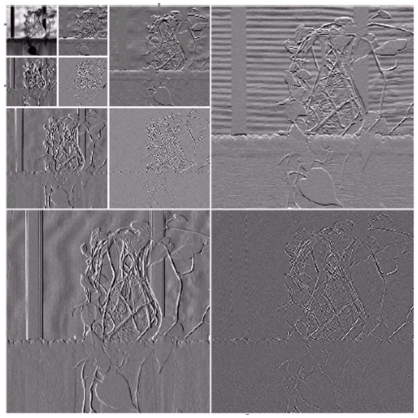
Filter banks and Haar transforms

- The only two-point FIR filters that satisfy the exact reconstruction conditions are the basis functions of the Haar transform.

$$h_0[n] = \frac{1}{\sqrt{2}}(\delta[n] + \delta[n-1])$$
$$h_1[n] = \frac{1}{\sqrt{2}}(\delta[n] - \delta[n-1])$$

- When these are used to form a wavelet system, the result is called a **Haar wavelet**.

Haar wavelet decomposition



- A filter bank is constructed from two-point FIR filters.
- The LL image is decomposed using the same filter bank.
- The LLLL image is also decomposed.
- Reconstructions at several resolutions are possible.

Multiresolution expansions

- In multiresolution analysis, a *scaling function*, is used to create a series of approximation of an image, each differing by a factor of 2 (in size) from its nearest neighboring approximations.
- Additional functions, called *wavelets*, are then used to encode the difference in information between adjacent approximations.

Series Expansions

- A signal or function *f*(*x*) can often be analyzed by expressing it as a linear combination of expansion functions

$$f(x) = \sum_k \alpha_k \varphi_k(x)$$

- If the set $\{\alpha_k\}$ is unique, the set of expansion functions form a *basis*.
- The set of expressible functions form a function space *V*.

$$V = \overline{span\{\varphi_k(x)\}}$$

- For any function space *V* and corresponding expansion set $\{\phi_i(x)\}$, there is a set of dual function $\{\tilde{\varphi}_k(x)\}$, that can be used to compute the α_i.

$$\alpha_k = \langle \tilde{\varphi}_k(x), f(x) \rangle = \int \tilde{\varphi}_k^*(x) f(x)\, dx$$

Orthogonal Bases and Frames

- Case 1:
  - If the expansion functions form an orthonormal basis, the basis and its dual are equivalent.
- Case 2:
  - If the expansion functions are merely orthogonal, the basis functions and their duals are *biorthogonal*.

$$\langle \varphi_j(x), \tilde{\varphi}_k(x) \rangle = \delta jk = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

- Case 3:
  - If the expansion set is not a basis, but supports the expansion, the expansion functions and their duals are *overcomplete* and form a *frame*.

Scaling functions

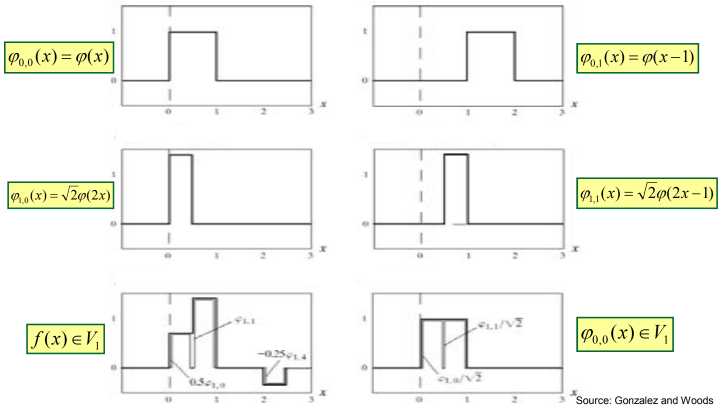
- Consider the set of expansion functions composed of integer translations and binary scalings of the real, square-integrable function $\varphi(x)$.

$$\varphi_{j,k}(x) = 2^{j/2} \varphi(2^j x - k)$$

- By choosing $\varphi(x)$ wisely, $\{\varphi_{j,k}(x)\}$ can be made to span $L^2(\mathbf{R})$.
- If we restrict *j*, the resulting expansion set will span a subset of $L^2(\mathbf{R})$.

$$V_j = \overline{\underset{k}{span}\{\varphi_{j,k}(x)\}}$$

Haar scaling functions

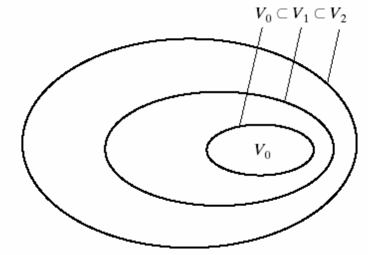


Source: Gonzalez and Woods

Requirements for multiresolution analysis (Mallat, 1989)

- #1: The scaling function is orthogonal to its integer translates.
- #2: The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales.
- #3: The only function that is common to all V_j is $f(x)=0$.
- #4: Any function can be represented with arbitrary precision.

The subspaces V_j are nested



source: Gonzalez and Woods

The dilation equation

- The expansion functions for subspace V_j can be expressed using the expansion functions for subspace V_{j+1}.

$$\varphi_{j,k}(x) = \sum_n \alpha_n \varphi_{j+1,n}(x)$$

- Changing from α_n to $h_\phi[n]$ and substituting

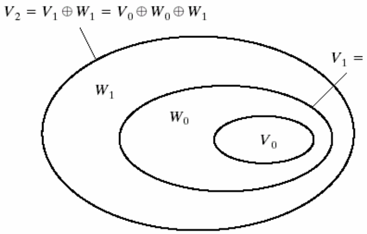
$$\varphi_{j,k}(x) = \sum_n h_\phi[n] 2^{(j+1)/2} \varphi(2^{j+1}x - n)$$

- Setting $j=k=0$

$$\boxed{\varphi(x) = \sum_n h_\phi[n]\sqrt{2}\,\varphi(2x-n)}$$

Wavelet functions

- Given a scaling function that meets the MRA requirements, we can define a *wavelet function* $\psi(x)$ that, together with its integer translates and binary scalings, spans the difference between any two adjacent scaling subspaces V_j and V_{j+1}.



$$\boxed{W_j = \underset{k}{span}\{\psi_{j,k}(x)\}}$$

W_i are orthogonal

Wavelets and scaling functions

- Since $W_j \subset V_j$, the wavelet can be expressed in terms of the scaling function.

$$\boxed{\psi(x) = \sum_n h_\psi[n]\sqrt{2}\varphi(2x-n)}$$

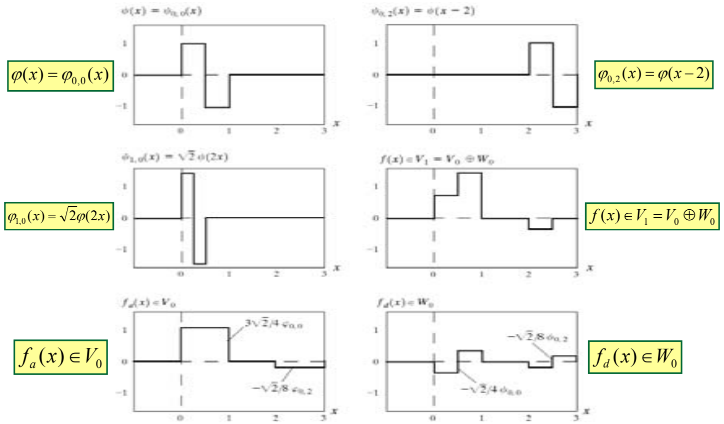
- It can be shown that

$$h_\psi[n] = (-1)^n h_\phi[1-n]$$

- In terms of filter banks,

$$h_0[n] = h_\phi[n] \qquad h_1[n] = h_\psi[n]$$

The Haar wavelet



$\varphi(x) = \varphi_{0,0}(x)$

$\varphi_{0,2}(x) = \varphi(x-2)$

$\varphi_{1,0}(x) = \sqrt{2}\varphi(2x)$

$f(x) \in V_1 = V_0 \oplus W_0$

$f_a(x) \in V_0$

$f_d(x) \in W_0$

1-D wavelet series expansion

- Any function $f(x) \in L^2(\boldsymbol{R})$ can be expanded relative to the wavelet $\psi(x)$ and scaling function $\varphi(x)$.

$$\boxed{f(x) = \sum_k c_{j_0}[k]\varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_k d_k[k]\psi_{j,k}(x)}$$

  - Analogous to Fourier series
- j_0 is an arbitrary starting scale
  - The $c_{j_0}[k]$ are normally called the ***approximation or scaling coefficients***.
  - The $d_k[k]$ are called the ***detail or wavelet coefficients***.

Calculating the wavelet series coefficients

- The wavelet series coefficients can be computed by performing the following inner products

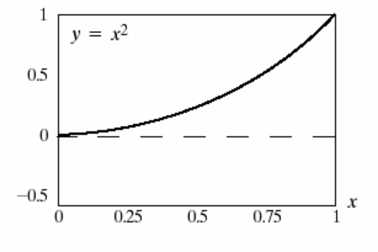
$$c_{j0}[k] = \langle f(x), \tilde{\varphi}_{jo,k}(x)\rangle = \int f(x)\tilde{\varphi}_{jo,k}(x)\,dx$$

$$d_j[k] = \langle f(x), \tilde{\psi}_{j,k}(x)\rangle = \int f(x)\tilde{\psi}_{j,k}(x)\,dx$$

Example

- Consider the simple example

$$y = \begin{cases} x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$



Grinding away

$$c_0(0) = \int_0^1 x^2 \varphi_{0,0}(x)\,dx = \int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$$d_0(0) = \int_0^1 x^2 \psi_{0,0}(x)\,dx = \int_0^{0.5} x^2\,dx - \int_{0.5}^1 x^2\,dx = -\frac{1}{4}$$

$$d_1(0) = \int_0^1 x^2 \psi_{1,0}(x)\,dx = \int_0^{0.25} x^2\sqrt{2}\,dx - \int_{0.25}^{0.5} x^2\sqrt{2}\,dx = -\frac{\sqrt{2}}{32}$$

$$d_1(1) = \int_0^1 x^2 \psi_{1,1}(x)\,dx = \int_{0.5}^{0.75} x^2\sqrt{2}\,dx - \int_{0.75}^{1} x^2\sqrt{2}\,dx = -\frac{3\sqrt{2}}{32}$$

The expansion

- Substituting these values gives the wavelet series expansion

$$y = \underbrace{\underbrace{\underbrace{\frac{1}{3}\varphi_{0,0}(x)}_{V_0} + \underbrace{\left[-\frac{1}{4}\psi_{0,0}(x)\right]}_{W_0}}_{V_1 = V_0 \oplus W_0} + \underbrace{\left[-\frac{\sqrt{2}}{32}\psi_{1,0}(x) - -\frac{3\sqrt{2}}{32}\psi_{1,1}(x)\right]}_{W_1}}_{V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1} + \cdots$$

Example at scales 0, 1, and 2