ECE6258 Lecture 24

Wavelet Expansions

Multiresolution expansions

- In multiresolution analysis, a *scaling function*, is used to create a series of approximation of an image, each differing by a factor of 2 (in size) from its nearest neighboring approximations.
- Additional functions, called *wavelets*, are then used to encode the difference in information between adjacent approximations.

Series Expansions

- A signal or function $f(x)$ can often be analyzed by expressing it as a linear combination of expansion functions

$$f(x) = \sum_k \alpha_k \varphi_k(x)$$

- If the set $\{\alpha_k\}$ is unique, the set of expansion functions form a *basis*.
- The set of expressible functions form a function space V.

$$V = \overline{span\{\varphi_k(x)\}}$$

- For any function space V and corresponding expansion set $\{\phi_i(x)\}$, there is a set of dual function $\{\tilde{\varphi}_k(x)\}$, that can be used to compute the α_i.

$$\alpha_k = \langle \tilde{\varphi}_k(x), f(x) \rangle = \int \tilde{\varphi}_k^*(x) f(x) dx$$

Scaling functions

- Consider the set of expansion functions composed of integer translations and binary scalings of the real, square-integrable function $\varphi(x)$.

$$\varphi_{j,k}(x) = 2^{j/2} \varphi(2^j x - k)$$

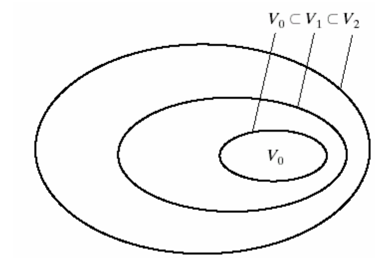
- By choosing $\varphi(x)$ wisely, $\{\varphi_{j,k}(x)\}$ can be made to span $L^2(\boldsymbol{R})$.
- If we restrict j, the resulting expansion set will span a subset of $L^2(\boldsymbol{R})$.

$$V_j = \overline{\underset{k}{span}\{\varphi_{j,k}(x)\}}$$

Requirements for multiresolution analysis (Mallat, 1989)

- #1: The scaling function is orthogonal to its integer translates.
- #2: The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales.
- #3: The only function that is common to all V_j is $f(x)=0$.
- #4: Any function can be represented with arbitrary precision.

The subspaces V_j are nested



source: Gonzalez and Woods

The dilation equation

- The expansion functions for subspace V_j can be expressed using the expansion functions for subspace V_{j+1}.

$$\varphi_{j,k}(x) = \sum_n \alpha_n \varphi_{j+1,n}(x)$$

- Changing from α_n to $h_\phi[n]$ and substituting

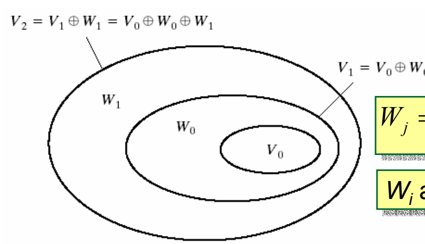
$$\varphi_{j,k}(x) = \sum_n h_\phi[n] 2^{(j+1)/2} \varphi(2^{j+1}x - n)$$

- Setting $j=k=0$

$$\varphi(x) = \sum_n h_\phi[n] \sqrt{2} \varphi(2x - n)$$

Wavelet functions

- Given a scaling function that meets the MRA requirements, we can define a *wavelet function* $\psi(x)$ that, together with its integer translates and binary scalings, spans the difference between any two adjacent scaling subspaces V_j and V_{j+1}.



$$W_j = \underset{k}{span}\{\psi_{j,k}(x)\}$$

W_i are orthogonal

Wavelets and scaling functions

- Since $W_j \subset V_j$, the wavelet can be expressed in terms of the scaling function.

$$\psi(x) = \sum_n h_\psi[n]\sqrt{2}\varphi(2x-n)$$

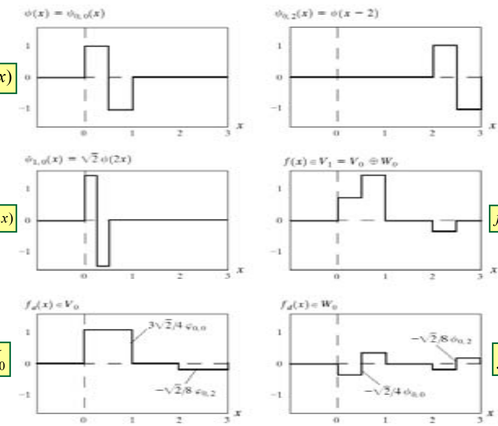
- It can be shown that

$$h_\psi[n] = (-1)^n h_\phi[1-n]$$

- In terms of filter banks,

$$h_0[n] = h_\phi[n] \qquad h_1[n] = h_\psi[n]$$

The Haar wavelet



1-D wavelet series expansion

- Any function $f(x) \in L^2(\boldsymbol{R})$ can be expanded relative to the wavelet $\psi(x)$ and scaling function $\varphi(x)$.

$$f(x) = \sum_k c_{j_0}[k]\varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_k d_k[k]\psi_{j,k}(x)$$

  - Analogous to Fourier series
- j_0 is an arbitrary starting scale
  - The $c_{j_0}[k]$ are normally called the ***approximation or scaling coefficients***.
  - The $d_k[k]$ are called the ***detail or wavelet coefficients***.

Calculating the wavelet series coefficients

- The wavelet series coefficients can be computed by performing the following inner products

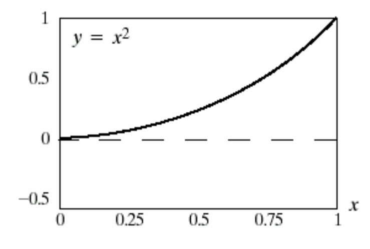
$$c_{j0}[k] = \langle f(x), \tilde{\varphi}_{jo,k}(x)\rangle = \int f(x)\tilde{\varphi}_{jo,k}(x)\,dx$$

$$d_j[k] = \langle f(x), \tilde{\psi}_{j,k}(x)\rangle = \int f(x)\tilde{\psi}_{j,k}(x)\,dx$$

Example

- Consider the simple example

$$y = \begin{cases} x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$



Grinding away

$$c_0(0) = \int_0^1 x^2 \varphi_{0,0}(x)\,dx = \int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$$d_0(0) = \int_0^1 x^2 \psi_{0,0}(x)\,dx = \int_0^{0.5} x^2\,dx - \int_{0.5}^{1} x^2\,dx = -\frac{1}{4}$$

$$d_1(0) = \int_0^1 x^2 \psi_{1,0}(x)\,dx = \int_0^{0.25} x^2\sqrt{2}\,dx - \int_{0.25}^{0.5} x^2\sqrt{2}\,dx = -\frac{\sqrt{2}}{32}$$

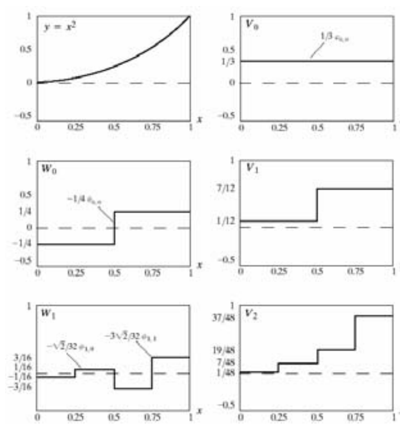
$$d_1(1) = \int_0^1 x^2 \psi_{1,1}(x)\,dx = \int_{0.5}^{0.75} x^2\sqrt{2}\,dx - \int_{0.75}^{1} x^2\sqrt{2}\,dx = -\frac{3\sqrt{2}}{32}$$

The expansion

- Substituting these values gives the wavelet series expansion

$$y = \underbrace{\underbrace{\underbrace{\frac{1}{3}\varphi_{0,0}(x)}_{V_0} + \underbrace{\left[-\frac{1}{4}\psi_{0,0}(x)\right]}_{W_0}}_{V_1 = V_0 \oplus W_0} + \underbrace{\left[-\frac{\sqrt{2}}{32}\psi_{1,0}(x) - -\frac{3\sqrt{2}}{32}\psi_{1,1}(x)\right]}_{W_1}}_{V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1} + \cdots$$

Example at scales 0, 1, and 2



Discrete wavelet transform

$$W_\psi[j,k] = \frac{1}{\sqrt{M}} \sum_n f[n] \tilde{\psi}_{j,k}[n]$$

$$W_\varphi[j_0,k] = \frac{1}{\sqrt{M}} \sum_n f[n] \tilde{\varphi}_{j_0,k}[n]$$

$$f[n] = \frac{1}{\sqrt{M}} \sum_k W_\varphi[j_0,k]\varphi_{j_0,k}[n] + \frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k W_\psi[j,k]\psi_{j,k}[n]$$

Fast wavelet transform

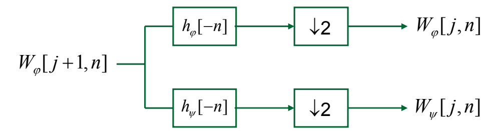
- Consider the discrete version of the dilation equation

$$W_\varphi[j,k] = \sum_m h_\varphi[m-2k] W_\varphi[j+1,m]$$

- And its counterpart for the wavelet function

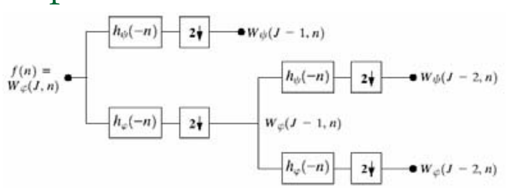
$$W_\psi[j,k] = \sum_m h_\psi[m-2k] W_\varphi[j+1,m]$$

- These suggest the following structure

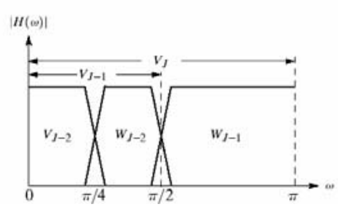


Two-band split

wavelet coefficient calculation

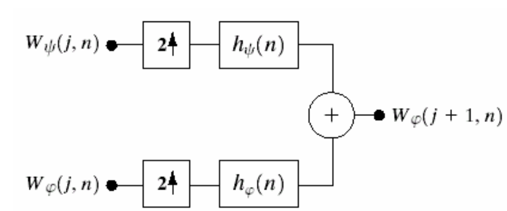


frequency characteristics of filters



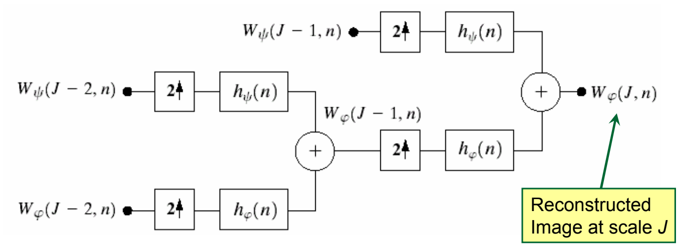
Fast inverse wavelet transform

- Using the analysis system and the known results for exact reconstruction filter banks give the synthesis system



Extending to more levels

- The whole series of computations can be iterated.



The FWT vs. the DFT

- For a transform of length $M = 2^J$…
  - Complexity of FWT is O(M).
  - Complexity of DFT (or DCT) is O(M log M).
- Existence
  - Depends upon vailability of a scaling function for a given wavelet.
  - Guaranteed for the DFT.
- Uncertainty
  - Using the DFT to get precise information about time, you must accept some vagueness about frequency and vice versa
  - Using the FWT we can isolate signal components in both frequency and time.

Time-frequency tilings