GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

ECE 6258
Digital Image Processing
Fall 2003

Problem Set #2

Issued: Wednesday, September 3, 2003
Due (live): Friday, September 12, 2003
Due (video): Friday, September 26, 2003

Problem 2.1 (Image Resizing): When magnifying an image, the goal is to produce a digital image y with support $\beta L \times \beta M$ pixels(with $\beta > 1$) from an image x with support $L \times M$. Since the pixel locations in the larger image, y, do not coincide with pixel locations in x, interpolations between the pixels of x is needed. Two simple techniques that can be used for this purpose are the following:

Nearest-Neighbor Interpolation: In this method the value of the new pixel S in image y is assigned as the value of the spatially closest pixel of x (when the inverse mapping is applied). Hence, S takes the value of one of S_1, S_2, S_3, or S_4, whichever is closest.

Bilinear Interpolation: The new pixel S in image y is computed from the four closest (inverse mapped) pixels S_1, S_2, S_3, and S_4 in image x.

$$S = (a)(b)S_1 + (1-a)(b)S_2 + (a)(1-b)S_3 + (1-a)(1-b)S_4$$

where

$$a = \frac{(S_2 - S)_h}{(S_2 - S_1)_h}$$
$$b = \frac{(S_3 - S)_v}{(S_3 - S_1)_v}$$

(a) For both techniques, answer the following: Is it linear? Is it shift-invariant?

(b) Show that both schemes are separable.

(c) We want to magnify a 4×4 image to size 8×8. Because of separability, we can write this as: $y = AxB$, where x is a 4×4 matrix, y is an 8×8 matrix and A and B are transposes of each other. Write A and B for both interpolation schemes.

(d) Implement your own MATLAB code for both schemes to magnify an image by an arbitrary factor $\beta > 1$. Magnify the CAMERAMAN image by a factor of 1.25. Turn in the magnified images.

(e) Compare the subjective quality of the magnified images and describe your observations.

Problem 1.2 (McClellan Transformations): Although procedures have been developed for the design of transformation functions, ad hoc methods often work well since the transformation typically involves very few free parameters. Ad hoc methods may take the form of specifying the mapping function for a few key frequencies. As an example, consider a first-order transformation of the form

$$F(\omega_1, \omega_2) = A + B\cos\omega_1 + C\cos\omega_2 + D\cos\omega_1\cos\omega_2$$

to design a non-separable lowpass filter with a rhomboidal passband that approximates the ideal response

$$H(\omega_1, \omega_2) = \begin{cases} 1, & |\omega_1| + |\omega_2| < \pi \\ 0, & \text{otherwise} \end{cases}$$

(a) Find a reasonable set of values for A, B, C, and D. Justify your answer. (You might consider such factors as: if using a lowpass prototype filter, what value of ω should map to the center of the 2-D passband? Where should $\omega = \pi$ map? What symmetries should the transformation function have? Can the transformation coefficients be constrained to have a constant value of the transformation function on the cutoff boundary of the ideal filter? Each question like these defines one or more linear equations that the transformation parameters must satisfy. If you get enough equations, F is determined.)

(b) Sketch the response of a 1-D prototype filter to be used with this transformation to design an approximation to this filter.

(c) The goal of this part of the problem is to design a 3-D zero-phase FIR filter using a McClellan transformation. The idealized passband of the filter should have the shape of an octahedron as shown in Figure 1. The cross-sectional views on selected planes of the frequency response, shown in Figure 2, define the orientation and cutoff frequencies of the passband. We propose to perform the design using a transformation of the form

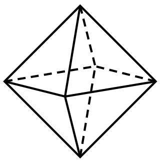

Figure 1: Ideal octahedral frequency response for Problem 2.2.

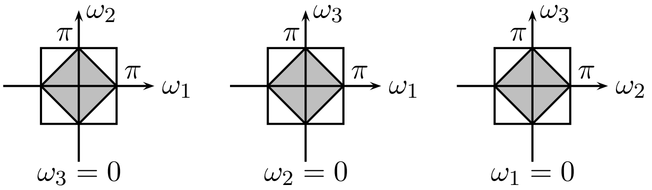

Figure 2: Cross-sections of octahedral frequency response.

$$\begin{aligned} F(\omega_1, \omega_2, \omega_3) &= A + B\cos\omega_1 + C\cos\omega_2 + D\cos\omega_3 + E\cos\omega_1\cos\omega_2 \\ &\quad + F\cos\omega_1\cos\omega_3 + G\cos\omega_2\cos\omega_3 + H\cos\omega_1\cos\omega_2\cos\omega_3 \end{aligned}$$

and a 1-D prototype filter that approximates the ideal behavior shown in Figure 3.

Solution:

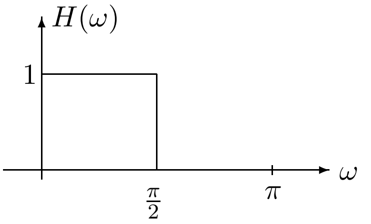

Figure 3: One-dimensional prototype filter for Problems 2.2.

(a) Assume that we will begin with a lowpass prototype filter. In order to map the center of the prototype passband ($\omega = 0$) to the center of the 2-D passband we might require that $\omega = 0$ map to $(\omega_1, \omega_2) = (0, 0)$ and similarly that $\omega = \pi$ map to $(\omega_1, \omega_2) = (\pi, \pi)$. Since $F(\omega_1, \omega_2) = \cos\omega$, these conditions say

$$A + B + C + D = 1$$
$$A - B - C + D = -1$$

We might also choose to require that the frequency response be constant on the line $\omega_1 = \pi - \omega_2$.

$$A + B\cos(\pi - \omega_2) + C\cos\omega_2 + D\cos(\pi - \omega_2)\omega_2 = const.$$

or

$$A - B\cos\omega_2 + C\cos\omega_2 - D\cos^2\omega_2 = const. \tag{1}$$

For this to be true, we must have $D = 0$ and $C = B$. Combining these two conditions with those from (1) and (1) gives the remaining parameters: A=0, B=0.5, and $C = 0.5$. The required transformation function is then

$$F(\omega_1, \omega_2) = 0.5(\cos\omega_1 + \cos\omega_2)\ .$$

(b) When $\omega_1 = \pi - \omega_2$, $F(\omega_1, \omega_2) = 0$, which implies that $\omega = \pi/2$. Therefore, the cutoff frequency of the 1-D lowpass prototype should be at $\pi/2$ radians. Then the cutoff frequency of the 2-D filter will fall on the indicated contours.

(c) (1) The basic transformation substitution is

$$\cos\omega = F(\omega_1, \omega_2, \omega_3)$$

Substituting $\omega = \omega_1 = \omega_2 = \omega_3 = 0$ gives the equation

$$A + B + C + D + E + F + G + H = 1.$$

(2) The constraint $F(\omega_1, \omega_2, 0) = F(\omega_1, \omega_2, 0)$ implies

$$B = C \qquad F = G.$$

The constraint $F(\omega_1, 0, \omega_3) = F(\omega_3, 0, \omega_1)$ implies

$$B = D \qquad E = G.$$

The constraint $F(0, \omega_2, \omega_3) = F(0, \omega_3, \omega_2)$ implies

$$C = D \qquad E = F.$$

These last two constraints, however, are implied by the earlier ones.

(3) Because of the symmetry constraints imposed in part (b), we only need to consider a single constraint equation. Using the one when $\omega_3 = 0$ and incorporating the above symmetry constraints

$$\begin{aligned}
const &= A + B\cos\omega_1 + B\cos(\pi - \omega_1) + B \\
&\quad + E\cos\omega_1\cos(\pi - \omega_1) + E\cos\omega_1\cos(\pi - \omega_1) \\
&\quad + E\cos\omega_1 + E\cos(\pi - \omega_1) + H\cos\omega_1\cos(\pi - \omega_1) \\
&= (A + B) + (0)\cos\omega_1 - (E + H)\cos^2\omega_1.
\end{aligned}$$

This equation implies the additional constraint

$$E = -H.$$

(4) At this point we have eight variables and six independent constraints. We can get a seventh by observing that the value of the constant in (3) should be zero since the outer boundary of the octahedron should correspond to $\omega = \pi/2$ and $\cos(\pi/2) = 0$. Therefore,

$$A = -B$$

A candidate for an eighth equation is to map $\omega = \pi$ to the corners of the cube $(-\pi, \pi) \times (-\pi, \pi) \times (-\pi, \pi)$.

$$\begin{aligned}
-1 &= F(\pi, \pi, \pi) \\
-1 &= A - 3B + 3E - H.
\end{aligned}$$

Solving these equations gives

$$\begin{aligned}
A &= -3/8; \qquad B = C = D = 3/8; \qquad E = F = G = 1/8; \\
H &= -1/8.
\end{aligned}$$

Problem 1.3 (Computing an Impulse Response): The ideal circular lowpass filter has the impulse response

$$h_c[n_1, n_2] = \frac{W}{2\pi} \frac{J_1(W\sqrt{n_1^2 + n_2^2})}{\sqrt{n_1^2 + n_2^2}}$$

and frequency response

$$H_c(\omega_1, \omega_2) = \begin{cases} 1, & \omega_1^2 + \omega_2^2 \leq W^2 < \pi^2 \\ 0, & \text{otherwise.} \end{cases}$$

Determine the impulse response of the ideal elliptical lowpass filter that has the frequency response

$$H_e(\omega_1, \omega_2) = \begin{cases} 1, & \frac{\omega_1^2}{a^2} + \frac{\omega_2^2}{b^2} \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

where $0 < a \leq b \leq \pi$.

Solution: Observe that $H_e(\omega_1, \omega_2)$ is simply a frequency scaled version of $H_c(\omega_1, \omega_2)$.

$$H_e(\omega_1, \omega_2) = H_c\left(\omega_1 \frac{W}{a}, \omega_2 \frac{W}{b}\right).$$

This substitution of variables can be applied to the inverse Fourier transform integral. Manipulating the integral into a form that is recognizable from the inverse Fourier transform of $H_c(\omega_1, \omega_2)$ allows us to write

$$\begin{aligned} h_e[n_1, n_2] &= \frac{ab}{W^2} \frac{W}{2\pi} \frac{J_1(W\sqrt{(\frac{a}{W}n_1)^2 + (\frac{b}{W}n_2)^2})}{\sqrt{(\frac{a}{W}n_1)^2 + (\frac{b}{W}n_2)^2}} \\ &= \frac{ab}{2\pi} \frac{J_1\left(\sqrt{(an_1)^2 + (bn_2)^2}\right)}{\sqrt{(an_1)^2 + (bn_2)^2}} \end{aligned}$$
