GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

Course ECE 2040

Circuit Analysis

February 25, 2000

Problem Set #6–Solutions

Problem 6.1: Express $v_{out}(t)$ in terms of $v_{in}(t)$ for the circuit in Figure 1.

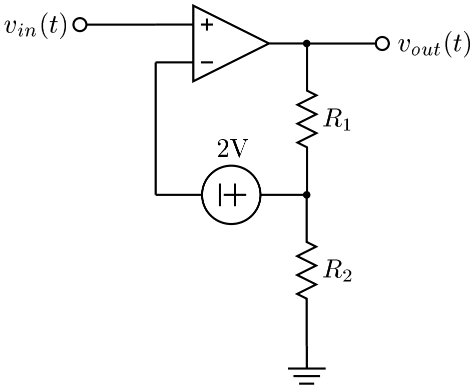

Figure 1: Circuit for Problem 6.1.

Solution: The node potential at the node where R_1, R_2, and the voltage source are connected is $v_{in}(t) + 2$. If we write a KCL equation at that node

$$\frac{v_{in}(t) + 2}{R_2} + \frac{v_{in}(t) + 2 - v_{out}(t)}{R_1} = 0.$$

Solving for $v_{out}(t)$ gives

$$v_{out}(t) = \left(1 + \frac{R_1}{R_2}\right)(v_{in}(t) + 2)$$

Problem 6.2: Find $v_o(t)$ for the circuit in Figure 2 in terms of the input voltages $v_a(t)$ and $v_b(t)$. The potentials at all of the terminals are measured with respect to a ground that is not shown.

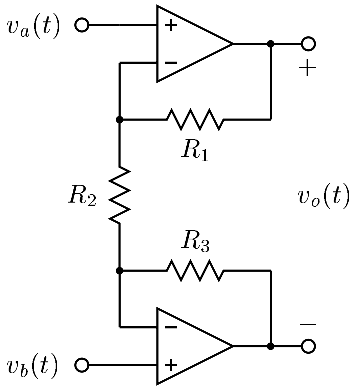

Figure 2: Circuit for Problem 6.2.

Solution: Let $v_+(t)$ be the node potential at the output of the upper opamp and $v_-(t)$ be the node potential at the output of the lower one. Then

$$v_o(t) = v_+(t) - v_-(t).$$

We can write KCL equations at the two nodes at opposite ends of R_2. The node potentials of these nodes are $v_a(t)$ and $v_b(t)$, respectively.

$$\begin{aligned}\text{upper node:}\quad & \frac{v_a(t) - v_+(t)}{R_1} + \frac{v_a(t) - v_b(t)}{R_2} = 0 \\ \text{lower node:}\quad & \frac{v_b(t) - v_-(t)}{R_3} + \frac{v_b(t) - v_a(t)}{R_2} = 0.\end{aligned}$$

From the first equation

$$\left[\frac{1}{R_1} + \frac{1}{R_2}\right] v_a(t) - \frac{v_+(t)}{R_1} - \frac{v_b(t)}{R_2} = 0.$$

If we solve this equation for $v_+(t)$ we get

$$v_+(t) = \left[1 + \frac{R_1}{R_2}\right] v_a(t) - \frac{R_1}{R_2} v_b(t).$$

If we use the second of the KCL equations to solve for $v_-(t)$ in a similar fashion, we get

$$v_-(t) = \left[1 + \frac{R_3}{R_2}\right] v_b(t) - \frac{R_3}{R_2} v_a(t).$$

Finally, we can subtract these two expressions.

$$v_o(t) = v_+(t) - v_-(t) = \left(1 + \frac{R_1}{R_2} + \frac{R_3}{R_2}\right)(v_a(t) - v_b(t))$$

This circuit therefore is a two opamp realization of a difference amplifier. Unlike the one opamp version, the output voltage is floating, i.e. it is not measured with respect to the ground.

Problem 6.3: Design a circuit containing a single operational amplifier that will produce an output voltage, $v_{out}(t)$ that is the derivative of the difference of two input voltages $v_a(t)$ and $v_b(t)$. Verify that your circuit works.

Solution: Consider the circuit in Figure 3. Remember that the current flowing through each capacitor is the derivative of the voltage drop across that capacitor. Let $e(t)$ be the node potential at each of the inputs to the opamp (these potentials are equal). Then the two KCL equations are:

$$\begin{aligned} C\frac{d}{dt}[e(t) - v_b(t)] + \frac{1}{R}[e(t) - v_{out}(t)] = 0 \\ C\frac{d}{dt}[e(t) - v_a(t)] + \frac{1}{R}e(t) = 0 \end{aligned}$$

From the second equation

$$\frac{1}{R}e(t) + C\frac{de(t)}{dt} = C\frac{dv_a(t)}{dt}$$

and from the first

$$\frac{1}{R}e(t) + C\frac{de(t)}{dt} = C\frac{dv_b(t)}{dt} + \frac{1}{R}v_{out}(t).$$

If we equate the two right-hand sides we get

$$\frac{1}{R}v_{out}(t) = C\left[\frac{dv_a(t)}{dt} - \frac{dv_b(t)}{dt}\right]$$

or

$$v_{out}(t) = RC\left[\frac{dv_a(t)}{dt} - \frac{dv_b(t)}{dt}\right]$$

We can control the gain of the integral by selecting appropriate values of R and C.

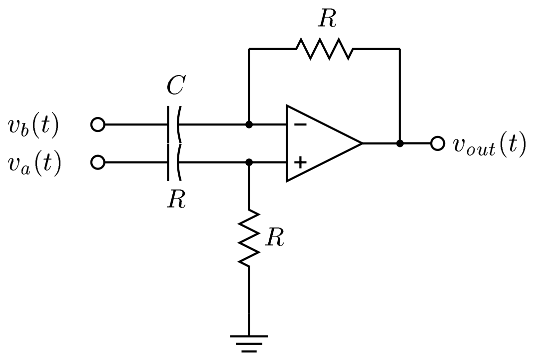

Figure 3: Solution to Problem 6.3.

Problem 6.4: Determine the Laplace transforms of the following time waveforms.

(a) $x_a(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$

(b) $x_b(t) = t^2 e^{-3t}, \;\; t > 0$

(c) $x_c(t) = e^{-4t} \sin 5t, \;\; t > 0$

(d) $x_d(t) = t, \;\; t > 0$

Solution:
(a)

$$X_a(s) = \int_0^T 1 \cdot e^{-st}\, dt = \frac{1}{s}\left(1 - e^{-sT}\right)$$

(b) To get $X_b(s)$ we can use an indirect approach (or we could just do the integral).

$$\begin{aligned} z(t) = e^{-3t} &\quad\longleftrightarrow\quad \frac{1}{s+3} \\ y(t) = te^{-3t} &\quad\longleftrightarrow\quad -\frac{d}{ds}\left(\frac{1}{s+3}\right) = \frac{1}{(s+3)^2} \\ x_b(t) = ty(t) &\quad\longleftrightarrow\quad -\frac{d}{ds}\left(\frac{1}{(s+3)^2}\right) = \frac{2}{(s+3)^3} \end{aligned}$$

(c)

$$\begin{aligned} x_c(t) &= e^{-4t}\sin 5t = \frac{1}{j2}\left(e^{-4t}e^{j5t} - e^{-4t}e^{-j5t}\right) \\ &= \frac{1}{j2}e^{-(4-j5)t} - \frac{1}{j2}e^{-(4+j5)t} \\ X_c(s) &= \frac{\frac{1}{j2}}{s+4-j5} - \frac{\frac{1}{j2}}{s+4+j5} \\ &= \frac{5}{s^2+8s+41} \end{aligned}$$

(d)

$$\begin{aligned} x_d(t) &= t = 1 \cdot t \\ X_d(s) &= -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2} \end{aligned}$$

Problem 6.5: Find the inverse Laplace transform of

$$X(s) = \frac{2s^2+6s+6}{s(s^2+5s+6)}.$$

Solution:

$$\begin{aligned} X(s) &= \frac{2s^2+6s+6}{s(s^2+5s+6)} = \frac{2s^2+6s+6}{s(s+2)(s+3)} \\ &= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3} \end{aligned}$$

We can evaluate A, B, and C using

$$\begin{aligned} A &= \lim_{s \to 0} \frac{2s^2+6s+6}{(s+2)(s+3)} = 1 \\ B &= \lim_{s \to -2} \frac{2s^2+6s+6}{s(s+3)} = -1 \\ C &= \lim_{s \to -3} \frac{2s^2+6s+6}{s(s+2)} = 2 \end{aligned}$$

Therefore,

$$x(t) = 1 - 2e^{-2t} + 2e^{-3t}, \;\; t > 0.$$