GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

**Course ECE 2040**

**Circuit Analysis**

March 3, 2000

## Problem Set #7–Solutions

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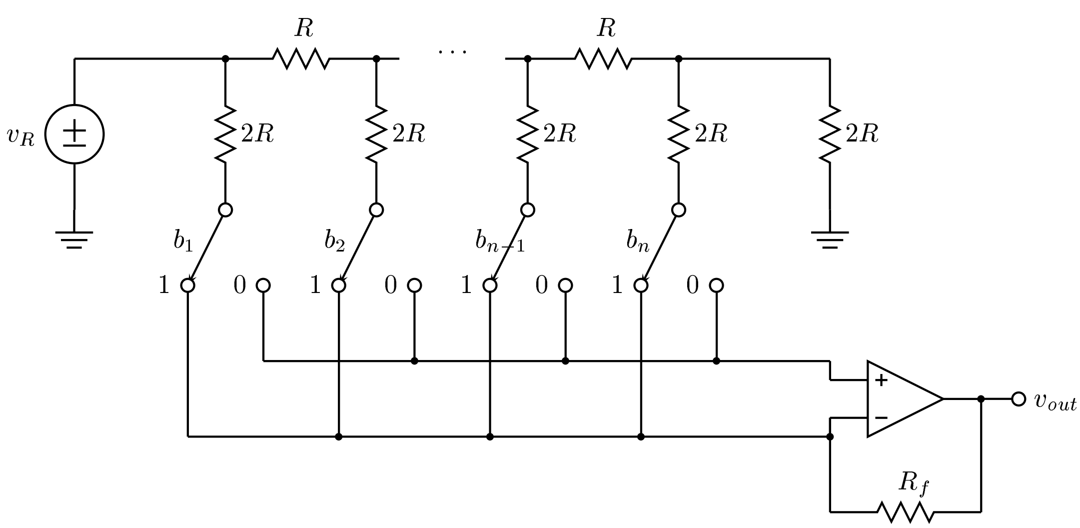
**Problem 7.1:** The circuit shown below is called the inverted $R - 2R$ ladder digital-to-analog converter (DAC). The input to this circuit is a binary code represented by $b_1$, $b_2$, …, $b_n$, where $b_i$ is either 1 or 0. Each switch shown in the figure is controlled by one of the bits in the binary code. If $b_i = 1$, that switch will be in the '1' position; if $b_i = 0$, that switch will be in the '0' position. Depending on the position of the switch, each current $i_k$ is diverted either to true ground (adding to the + terminal of the opamp) or to the virtual ground (adding to the − terminal.)

(a) If $i$ is the current flowing out of the voltage source, show that $i = v_R/R$ regardless of the digital input code.

(b) Show that the output voltage can be expressed as

$$v_{out}(t) = -\frac{R_f}{R}v_R(b_1 2^{-1} + b_2 2^{-2} + \ldots + b_{n-1}2^{-n+1} + b_n 2^{-n})$$

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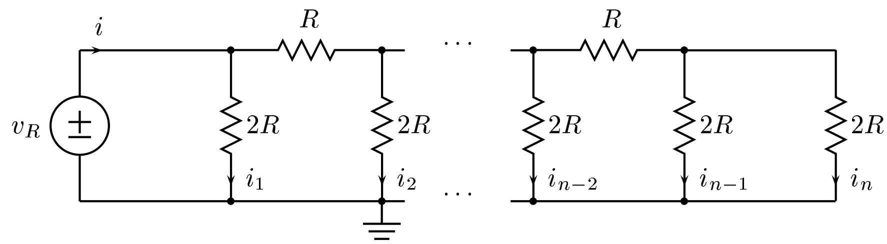
**Solution:**

(a) The voltage source see the ladder of resistors. Because the voltages at the two input terminals of the opamp are virtually equal to each other, all of the $2R$ resistors are essentially connected to ground. Thus from the point-of-view of the voltage source the circuit looks like




This is readily seen to be equivalent to a resistance of $R\Omega$. Thus, $i = v/R$. Using current dividers

$$\begin{aligned} i_1 &= \frac{i}{2} = \frac{v}{2R} \\ i_2 &= \frac{i}{2^2} = \frac{v}{2^2 R} \\ &\vdots \\ i_n &= \frac{i}{2^n} = \frac{v}{2^n R} \end{aligned}$$

(b)

$$\frac{v_{out}(t)}{R_f} = b_1 i_1 + b_2 i_2 + \ldots + b_n i_n$$

or

$$v_{out}(t) = \frac{R_f v_R}{R}\left[b_1 2^{-1} + b_2 2^{-2} + \ldots + b_n 2^{-n}\right].$$

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**Problem 7.2:** Find the inverse Laplace transforms (for $t \geq 0$) for the following functions:

(a) $X_a(s) = \frac{s^2}{(s+1)^2+1}$

(b) $X_b(s) = \frac{s+1}{s^2(s+2)}$

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**Solution:**

(a)

$$\begin{aligned} X_a(s) &= \frac{s^2}{(s+1)^2+1} \\ &= 1 + \frac{A}{s+1+j} + \frac{A^*}{s+1-j} \end{aligned}$$

$$A = \lim_{s \to -1-j}\left[\frac{s^2}{s+1-j}\right] = 1$$

Therefore,

$$X_a(s) = 1 + \frac{1}{s+1+j} + \frac{1}{s+1-j}$$

and

$$\begin{aligned} x_a(t) &= \delta(t) + e^{-t}e^{-jt} + e^{-t}e^{jt} \\ &= \delta(t) + 2e^{-t}\cos t, \quad t \geq 0. \end{aligned}$$

(b)

$$\begin{aligned} X_b(s) &= \frac{s+1}{s^2(s+2)} \\ &= \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+2} \end{aligned}$$

We can get $A$ and $C$ straightforwardly.

$$\begin{aligned} A &= \lim_{s\to 0} \frac{s+1}{s+2} = \frac{1}{2} \\ C &= \lim_{s\to -2} \frac{s+1}{s^2} = -\frac{1}{4} \end{aligned}$$

Therefore,

$$X_b(s) = \frac{1/2}{s^2} + \frac{B}{s} - \frac{1/4}{s+2}$$

To determine $B$, we can recombine this into a single fraction

$$X_b(s) = \frac{(B - \frac{1}{4})s^2 + (\frac{1}{2} + 2B)s + 1}{s^2(s+2)}$$

from which we see that $B = \frac{1}{4}$. Therefore,

$$X_b(s) = \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s} + \frac{\frac{1}{4}}{s+2}$$

and

$$x_b(t) = \frac{1}{2}t + \frac{1}{4} - \frac{1}{4}e^{-2t}, \quad t \geq 0.$$

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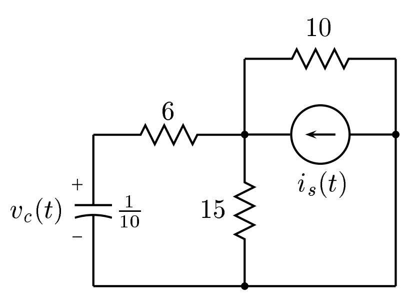
**Problem 7.3:** For the circuit below, determine $v_c(t)$ if $i_s(t) = u(t)$. Assume that $v_c(0) = 0$.

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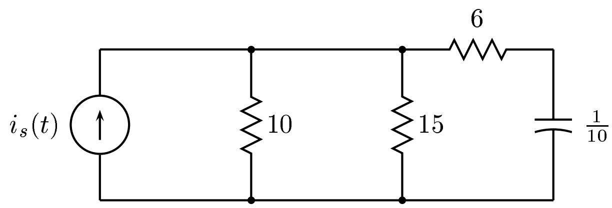
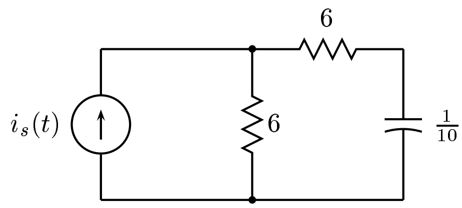
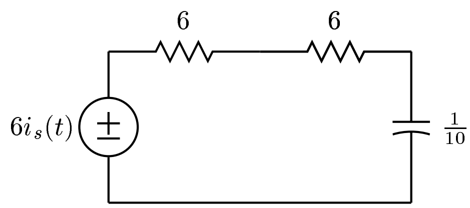
**Solution:** As an alternative to setting up and solving a set of equations to determine $V_c(s)$, we can simplify the circuit by replacing the current source and resistors by their Thevenin equivalent. First, we redraw the circuit to give it a more familiar appearance



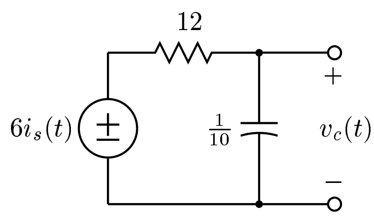

After combining the parallel resistors this becomes




Now we can replace the current source in parallel with a resistor with a resistor in series with a voltage source. This gives




Finally, this becomes




Converting to the Laplace domain and applying a voltage divider

$$V_c(s) = \frac{\frac{10}{s}}{12 + \frac{10}{s}} \cdot \frac{6}{s} = \frac{5}{s(s + \frac{5}{6})}$$
$$= \frac{A}{s} + \frac{B}{s + \frac{5}{6}}$$

$$A = \lim_{s \to 0} \frac{5}{s + \frac{5}{6}} = 6$$
$$B = \lim_{s \to -\frac{5}{6}} \frac{5}{s} = -6$$

Therefore,

$$v_c(t) = 6(1 - e^{-5t/6}), \quad t > 0.$$

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**Problem 7.4:** For the circuit in Figure 1, let $i_s(t) = 1$ for $t > 0$ and assume that at $t = 0$ the current through the inductor is zero and the voltage drop on the capacitor is 1 volt.

(a) Redraw the circuit in the Laplace domain.

(b) Determine $V(s)$, the Laplace transform of the resistor voltage.

(c) Determine $v(t)$ for $t > 0$.

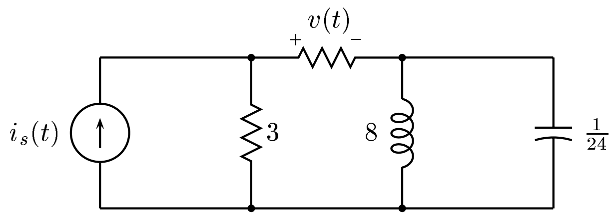


Figure 1: Circuit for Problem 7.4.

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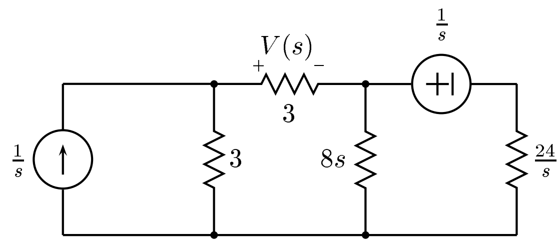
**Solution:**

(a) We replace the inductor and capacitor by impedances and insert a voltage source to realize the initial voltage on the capacitor.




(b) Let $\frac{1}{s}$, $I_\alpha(s)$, and $I_\beta(s)$ be mesh currents in the three meshes. Then, if we write KVL equations over the two rightmost meshes, we get

$$3[I_\alpha(s) - \frac{1}{s}] + 3I_\alpha(s) + 8s[I_\alpha(s) - I_\beta(s)] = 0$$
$$8s[I_\beta(s) - I_\alpha(s)] + \frac{24}{s} I_\beta(s) = -\frac{1}{s}.$$

We can rewrite these as

$$[8s + 6]I_\alpha(s) - 8sI_\beta(s) = \frac{3}{s}$$
$$-8sI_\alpha(s) + [8s + \frac{24}{s}]I_\beta(s) = -\frac{1}{s}$$

If we add the two equations together we get

$$6I_\alpha(s) + \frac{24}{s} I_\beta(s) = \frac{1}{s}$$

from which

$$I_\beta(s) = -\frac{s}{4} I_\alpha(s) + \frac{1}{3s}$$

If we substitute this result into the first equation, we get

$$I_\alpha(s) = \frac{\frac{4}{3}s + \frac{3}{2}}{s(s^2 + 4s + 3)}$$

Finally,

$$V(s) = 3I_\alpha(s) = \frac{4s + \frac{9}{2}}{s(s^2 + 4s + 3)}$$

(c) $V(s) = \frac{\frac{3}{2}}{s} + \frac{-\frac{1}{4}}{s+1} + \frac{-\frac{5}{4}}{s+3}$. Therefore,

$$v(t) = \frac{3}{2} - \frac{1}{4}e^{-t} - \frac{5}{4}e^{-3t}, \;\; t > 0$$

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**Problem 7.5:** The circuit in Figure 2 is at initial rest. This means that at $t = 0$ there is no current flowing through the inductor and no voltage drop across the capacitor. Determine $v(t)$ for all $t$ if $i_s(t) = 2e^{-3t}u(t)$.

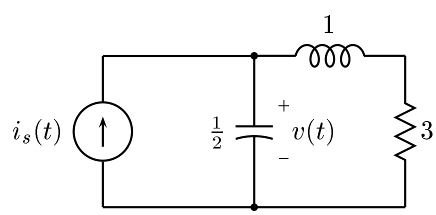


Figure 2: Circuit for Problem 7.5.

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**Solution:** We begin be mapping the circuit to the Laplace domain. The inductor is replaced by an impedance with a value of $s$, the capacitor is replaced by an impedance with a value of $2/s$ and the current source value is

$$I_s(s) = \frac{2}{s+3}.$$

Since the circuit is at initial rest, there is no need to introduce auxiliary sources to accommodate the initial values.

The capacitor voltage can be found from the equivalent impedance of the circuit seen by the current source.

$$V(s) = Z_{eq}(s)I_s(s) = \frac{\frac{2}{s}Z_{r\ell}(s)}{\frac{2}{s}Z_{r\ell}(s)}I_s(s)$$

$Z_{r\ell}(s)$ is the equivalent impedance of the series $RL$.

$$Z_{r\ell}(s) = s + 3$$

Therefore, if we make the appropriate substitution we get

$$\begin{aligned} V(s) &= \frac{4}{s^2 + 3s + 2} \\ &= \frac{4}{s+1} - \frac{4}{s+2}. \end{aligned}$$

After computing an inverse Laplace transform we finally arrive at the answer

$$v(t) = 4e^{-t} - 4e^{-2t}, \quad t \geq 0$$