GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

Course EE 2250

Electric Circuit Analysis

January 28, 1999

Problem Set #3—Solutions

Problem 3.1: (a) Show that the network in Figure 1 is equivalent to a single resistor.

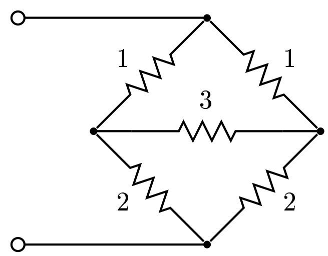
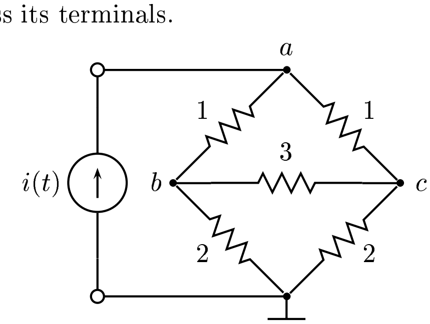


Figure 1:

(b) Determine the value of the equivalent resistance, R_{eq}.

Solution: The quickest solution to this problem would be to appeal to the summetry of the circuit to argue that no current could flow through the 3Ω resistor. If no current flows through it, it can be removed. The one-port can then be simplified using series and parallel combinations. This approach will not work, however, it the circuit is not balanced, so it is also appropriate to analyze the circuit without this shortcut.

(a) We can find the equivalent circuit by attaching a current source and finding the voltage drop across its terminals.



To analyze the resulting circuit we apply the node method. Let us choose to write KCL equations at nodes a, b, and c. Observe that the potential at node a

is equal to $v(t)$, the voltage that we seek.

$$\begin{aligned}
\text{node } a: &\quad [e_a(t) - e_b(t)] + [e_a(t) - e_c(t)] = i(t) \\
\text{node } b: &\quad [e_b(t) - e_a(t)] + \frac{1}{2}e_b(t) + \frac{1}{3}[e_b(t) - e_c(t)] = 0 \\
\text{node } c: &\quad [e_c(t) - e_a(t)] + \frac{1}{2}e_c(t) + \frac{1}{3}[e_c(t) - e_b(t)] = 0
\end{aligned}$$

These reduce to

$$\begin{aligned}
2e_a(t) - e_b(t) - e_c(t) &= i(t) \\
-e_a(t) + \frac{11}{6}e_b(t) - \frac{1}{3}e_c(t) &= 0 \\
-e_a(t) - \frac{1}{3}e_b(t) + \frac{11}{6}e_c(t) &= 0
\end{aligned}$$

Subtracting the last two equations from each other establishes that

$$e_b(t) = e_c(t).$$

Substituting this fact into the first two equations gives

$$\begin{aligned}
e_a(t) - e_b(t) &= \frac{1}{2}i(t) \\
-e_a(t) + \frac{3}{2}e_b(t) &= 0
\end{aligned}$$

from which

$$e_a(t) = \frac{3}{2}i(t).$$

Since the voltage is proportional to the current, the circuit is equivalent to a resistor.

(b) The equivalent resistance is $\frac{3}{2}\Omega$.

Problem 3.2: (a) Determine the equivalent resistance of the two-terminal network in Figure 2a as seen at the terminals $a - a'$.

(b) Repeat for the network in Figure 2b at the terminals $b - b'$.

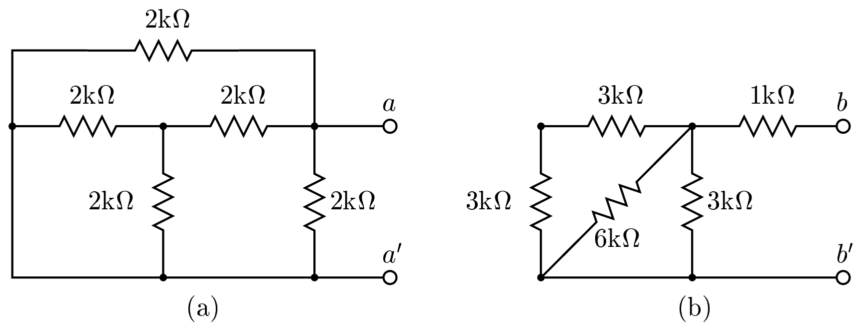
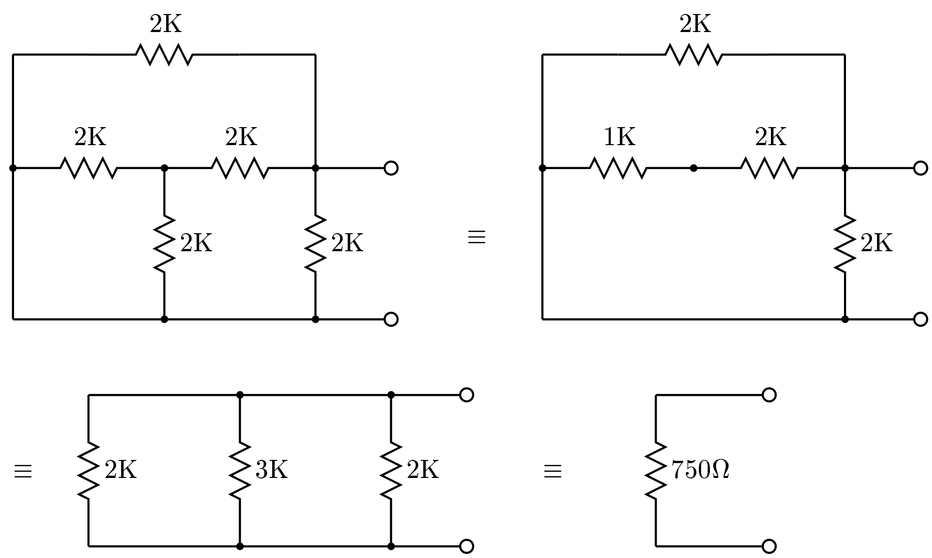


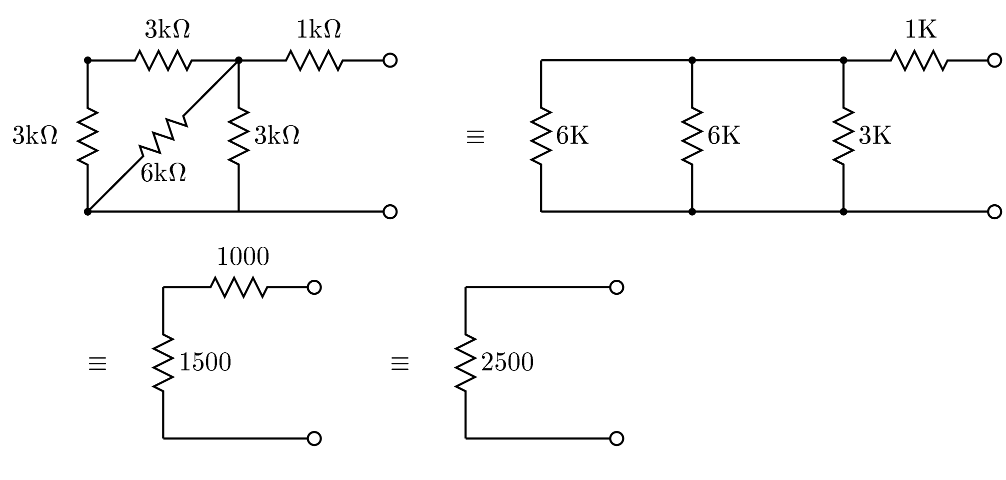
Figure 2:

Solution:

(a)



(b)



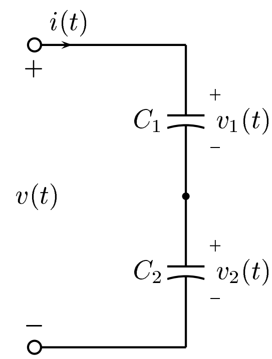
Problem 3.3: Consider a one-port network consisting of two capacitors with capacitances C_1 and C_2 connected in series.

(a) Show that this network is equivalent to a single capacitor.

(b) Derive a formula for the equivalent capacitance C_{eq} in terms of C_1 and C_2.

(c) Derive expressions for the voltage $v_1(t)$ measured across capacitor C_1 and the voltage $v_2(t)$ measured across C_2 in terms of the voltage $v(t)$ appearing across the series connection.

Solution:



$$\begin{aligned} v(t) &= v_1(t) + v_2(t) \\ &= \frac{1}{C_1}\int_{-\infty}^{t} i(\beta)\,d\beta + \frac{1}{C_2}\int_{-\infty}^{t} i(\beta)\,d\beta \\ &= \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int_{-\infty}^{t} i(\beta)\,d\beta. \end{aligned}$$

(a) This is the $v - i$ relation for a capacitor.

(b)

$$\frac{1}{C_{\text{eq}}} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

which implies

$$C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2}$$

(c)

$$\frac{v_1(t)}{v(t)} = \frac{\frac{1}{C_1}\int_{-\infty}^{t} i(\beta)\,d\beta}{\left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int_{-\infty}^{t} i(\beta)\,d\beta} = \frac{\frac{1}{C_1}}{\frac{1}{C_1} + \frac{1}{C_2}}$$

Therefore,

$$v_1(t) = \frac{C_2}{C_1 + C_2} v(t).$$

Similarly,

$$v_2(t) = \frac{C_1}{C_1 + C_2} v(t).$$

Problem 3.4: In an attempt to determine a Thévenin equivalent for a network containing only resistors and sources two experiments are performed. First, a resistor with a value of R ohms is connected across the terminals and the voltage $v_1(t)$ is measured, as shown in Figure 3. Then a resistor with a value of $2R$ ohms is connected across the terminals and the voltage $v_2(t)$ is measured.

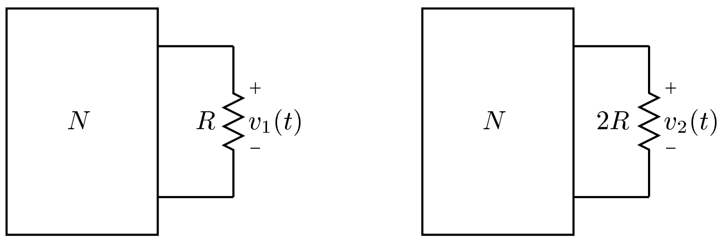
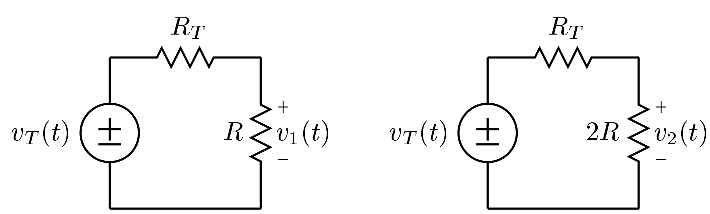


Figure 3:

(a) Determine the Thévenin equivalent model for the network in terms of the measured voltages $v_1(t)$ and $v_2(t)$.

(b) What is a good value to choose for R? Explain your answer.

Solution:



(a) We begin be replacing the network by its Thevenin equivalent as in the previous figure. The goal of the problem is to determine R_T and $v_T(t)$. For Experiment #1

$$v_1(t) = \frac{R}{R + R_T} v_T(t)$$

and for Experiment #2

$$v_2(t) = \frac{2R}{2R + R_T} v_T(t)$$

This gives us two equations in two unknowns that we can solve for R_T and $v_T(t)$.

$$\begin{aligned} Rv_1(t) + R_T v_1(t) &= Rv_T(t) \\ 2Rv_2(t) + R_T v_2(t) &= 2Rv_T(t) \end{aligned}$$

The solution to these is

$$R_T = -\frac{2R(v_1(t) - v_2(t))}{2v_1(t) - v_2(t)}$$
$$v_T(t) = v_1(t)\left[1 + \frac{2(v_1(t) - v_2(t))}{2v_1(t) - v_2(t)}\right]$$

Since $v_1(t)$ and $v_2(t)$ are both proportional to $v_T(t)$, R_T and the quantity between the brackets in the expression for $v_T(t)$ are both constants.

(b) There are no hard and fast rules here, but it would be good if $R \approx R_T$ so that $v_1(t)$ and $v_2(t)$ are significantly different in amplitude to reduce the sensitivity to measurement errors.

Problem 3.5: Consider the two two-terminal networks N_1 and N_2 shown in Figure 4. The two

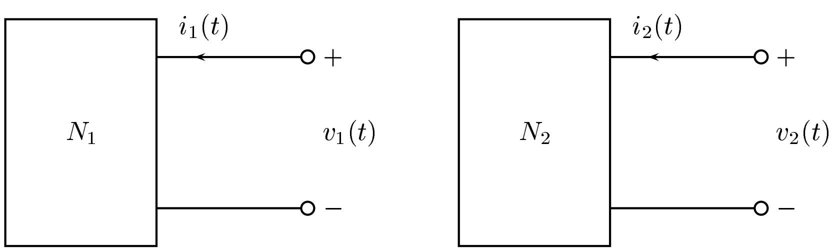


Figure 4:

networks have the $v - i$ relations

$$N_1: \quad v_1(t) = 4i_1(t) - 8$$
$$N_2: \quad v_2(t) = 2i_2(t) + 3$$

(a) Determine the equilibrium values of $v(t)$ and $i(t)$ if the two netorks are connected as shown in Figure 5.

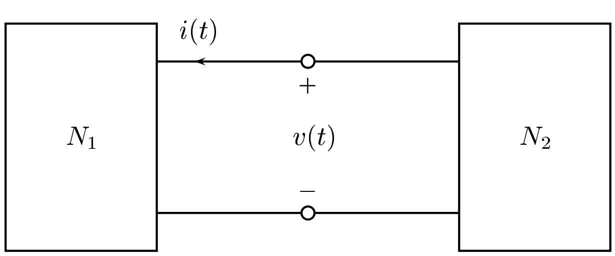


Figure 5:

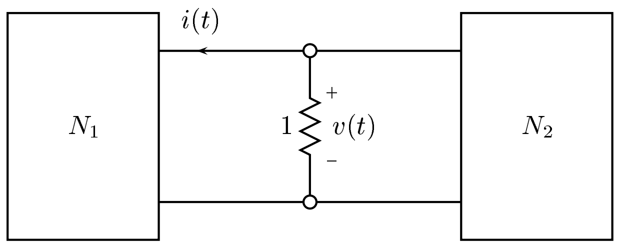


Figure 6:

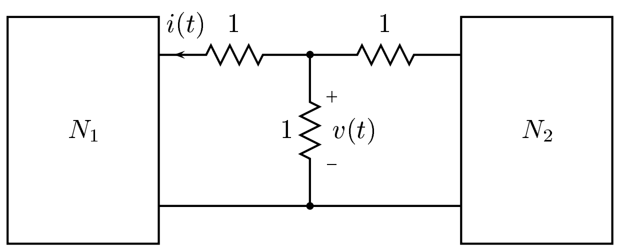
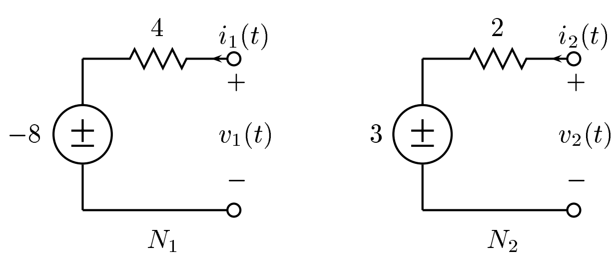


Figure 7:

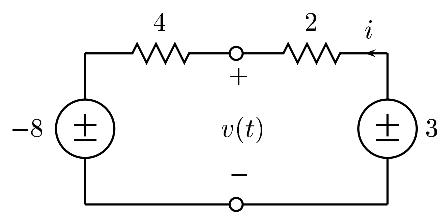
(b) Repeat part (a) for the connection shown in Figure 6.

(c) Repeat part (a) for the connection shown in Figure 7.

Solution: The easiest approach here is to replace each network by its Thevenin equivalent. Then all three parts of the problem become quite straightforward. The two Thevenin equivalent circuits are:



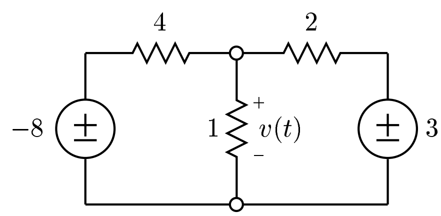
(a) Inserting the Thevenin equivalents, the circuit becomes



The current i is quickly seen to be equal to $11/6$A, so that

$$v = 3 - 2 \cdot \frac{11}{6} = -\frac{2}{3}V.$$

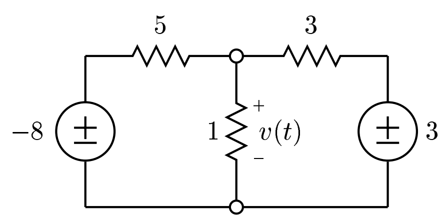
(b) Here the equivalent circuit is seen to be



Let the potential at the upper node where the three resistors are joined by e. Then, if we write a KCL equation at that node we get

$$\frac{1}{4}(e+8) + \frac{1}{2}(e-3) + e = 0$$
$$\frac{7}{4}e = -\frac{1}{2}$$
$$e = -\frac{2}{7}V.$$

(c) Two of the 1Ω resistors are in series with the Thevenin equivalent resistors. Once these are combined with those resistors, the resulting circuit looks identical to the one in part (b), expect for the change in the resistor values.



Now

$$\frac{1}{5}(e+8) + \frac{1}{3}(e-3) + e = 0$$
$$\frac{23}{15}e = -\frac{3}{5}$$
$$e = -\frac{9}{23}V.$$

Problem 3.6: Design a circuit with four inputs $v_1(t)$, $v_2(t)$, $v_3(t)$ and $v_4(t)$ containing a single operational amplifier, such that the output voltage $v_{out}(t)$ satisfies

$$v_{out}(t) = k_5[k_1 v_1(t) + k_2 v_2(t) - k_3 v_3(t) - k_4 v_4(t)]$$

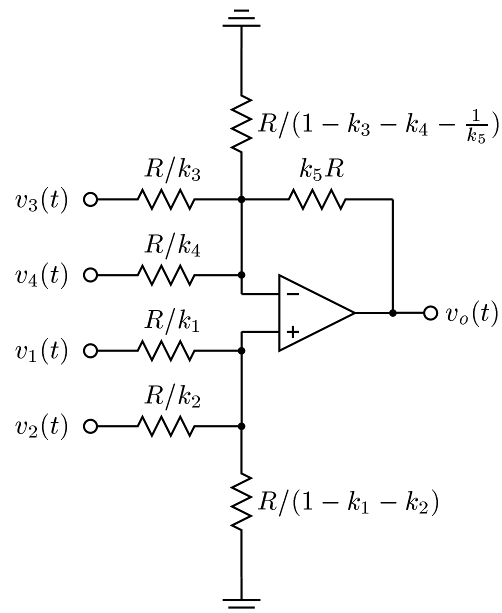
for positive values of the four constants. *Hint:* You might begin by trying to combine features of the differential amplifier configuration and the summing amplifier configuration.

(b) Prove that your design in (a) works correctly.

(c) State any additional constraints that need to be imposed on the four constants.

Solution: The approach with this problem is to combine the structures for the differential amplifier and the summing amplifier and then through trial and error to work out a set of values for the resistors that will give the desired input-output relation. It is easiest if all of the resistors are proportional to some base resistance R.

(a) One circuit that will work is the following



(b) Let the potential at both the inverting and non-inverting inputs of the opamp be $e(t)$. Then from KCL at the non-inverting node:

$$\frac{e(t)-v_1(t)}{R/k_1}+\frac{e(t)-v_2(t)}{R/k_2}+\frac{e(t)}{R/(1-k_1-k_2)}=0.$$

This equation reduces to

$$e(t)=k_1v_1(t)+k_2v_2(t). \tag{1}$$

From KCL at the inverting node

$$\frac{e(t)-v_3(t)}{R/k_3}+\frac{e(t)-v_4(t)}{R/k_4}+\frac{e(t)-v_0(t)}{k_5R}+\frac{e(t)-v_1(t)}{R/(1-k_3-k_4-\frac{1}{k_5})}=0$$

This one reduces to

$$e(t)=\frac{1}{k_5}v_o(t)+k_4v_4(t)+k_3v_3(t) \tag{2}$$

Equating these two equations for $e(t)$ ((1) and (2))gives

$$\frac{1}{k_5}v_o(t)+k_4v_4(t)+k_3v_3(t)=k_1v_1(t)+k_2v_2(t)$$

or

$$v_o(t)=k_5[k_1v_1(t)+k_2v_2(t)-k_3v_3(t)-k_4v_4(t)]$$

(c) All of the resistor values must be positive. Thus, we must have

$$k_5 > 1$$
$$k_1 > 0;\; k_2 > 0;\; k_3 > 0;\; k_4 > 0$$
$$k_1 + k_2 < 1$$
$$k_3 + k_4 + \frac{1}{k_5} < 1$$
