GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

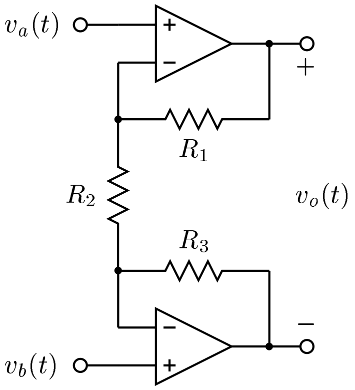
Course EE 2250

Electric Circuit Analysis

February 4, 1999

Problem Set #4–Solutions

Problem 4.1: Find $v_o(t)$ for the circuit below in terms of the input voltages $v_a(t)$ and $v_b(t)$. The potentials at all of the terminals are measured with respect to a ground that is not shown.



Solution: Let $v_+(t)$ be the node potential at the output of the upper opamp and $v_-(t)$ be the node potential at the output of the lower one. Then

$$v_o(t) = v_+(t) - v_-(t).$$

We can write KCL equations at the two nodes at opposite ends of R_2. The node potentials of these nodes are $v_a(t)$ and $v_b(t)$, respectively.

$$\text{upper node:} \quad \frac{v_a(t) - v_+(t)}{R_1} + \frac{v_a(t) - v_b(t)}{R_2} = 0$$

$$\text{lower node:} \quad \frac{v_b(t) - v_-(t)}{R_3} + \frac{v_b(t) - v_a(t)}{R_2} = 0.$$

From the first equation

$$\left[\frac{1}{R_1} + \frac{1}{R_2}\right] v_a(t) - \frac{v_+(t)}{R_1} - \frac{v_b(t)}{R_2} = 0.$$

If we solve this equation for $v_+(t)$ we get

$$v_+(t) = \left[1 + \frac{R_1}{R_2}\right] v_a(t) - \frac{R_1}{R_2} v_b(t).$$

If we use the second of the KCL equations to solve for $v_-(t)$ in a similar fashion, we get

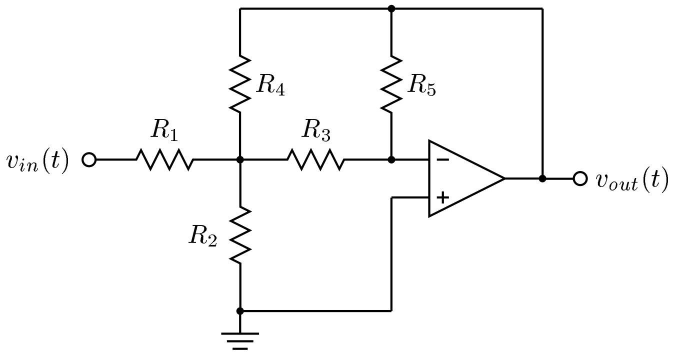
$$v_-(t) = \left[1 + \frac{R_3}{R_2}\right] v_b(t) - \frac{R_3}{R_2} v_a(t).$$

Finally, we can subtract these two expressions.

$$v_o(t) = v_+(t) - v_-(t) = \left(1 + \frac{R_1}{R_2} + \frac{R_3}{R_2}\right)(v_a(t) - v_b(t))$$

This circuit therefore is a two opamp realization of a difference amplifier. Unlike the one opamp version, the output voltage is floating, i.e. it is not measured with respect to the ground.

Problem 4.2: Determine the output voltage $v_{out}(t)$ in terms of the input voltage $v_{in}(t)$ for the circuit below.



Solution: We can write KCL equations at the two nodes on the opposite ends of R_3. Let the potential at the left node by $e(t)$. The potential at the right node is zero.

$$\text{left node:} \quad \frac{e(t) - v_{in}(t)}{R_1} + \frac{e(t) - v_{out}(t)}{R_4} + \frac{e(t)}{R_3} + \frac{e(t)}{R_2} = 0$$

$$\text{right node:} \quad -\frac{e(t)}{R_3} - \frac{v_{out}(t)}{R_5} = 0$$

The first of these equations reduces to

$$e(t)\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right] = \frac{1}{R_1} v_{in}(t) + \frac{1}{R_4} v_{out}(t)$$

and the second reduces to

$$e(t) = -\frac{R_3}{R_5} v_{out}(t).$$

We can substitute the latter expression into the earlier equation and solve for $v_{out}(t)$ to get

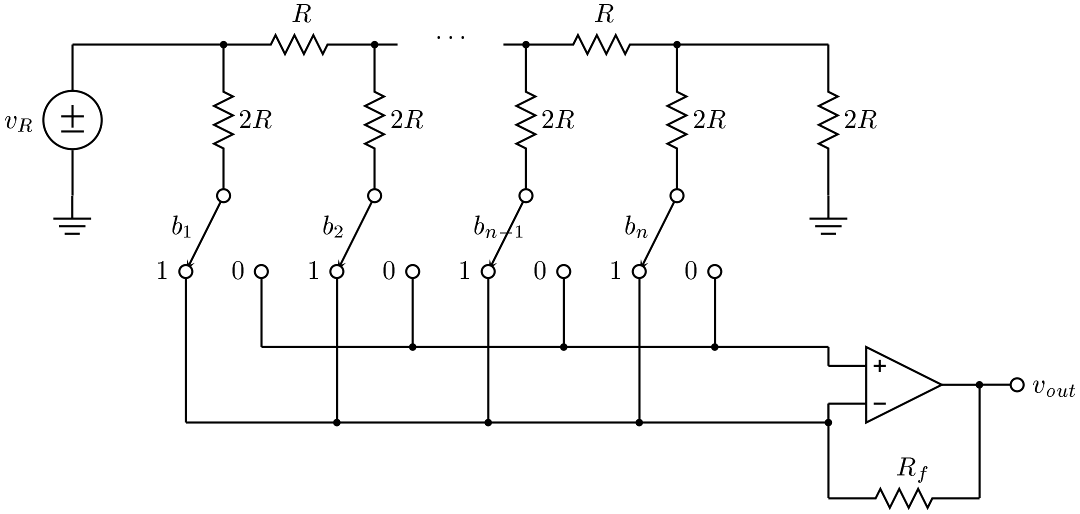
$$v_{out}(t) = \frac{\frac{-1}{R_3 R_4}}{\frac{1}{R_5}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) + \frac{1}{R_3 R_4}} v_{in}(t).$$

Problem 4.3: This is Problem 6.6-16 from Dorf and Svoboda. The circuit shown below is called the inverted $R-2R$ ladder digital-to-analog converter (DAC). The input to this circuit is a binary code represented by b_1, b_2, …, b_n, where b_i is either 1 or 0. Each switch shown in the figure is controlled by one of the bits in the binary code. If $b_i = 1$, that switch will be in the '1' position; if $b_i = 0$, that switch will be in the '0' position. Depending on the position of the switch, each current i_k is diverted either to true ground (adding to the + terminal of the opamp) or to the virtual ground (adding to the − terminal.)

(a) Show that $i = v_R/R$ regardless of the digital input code.

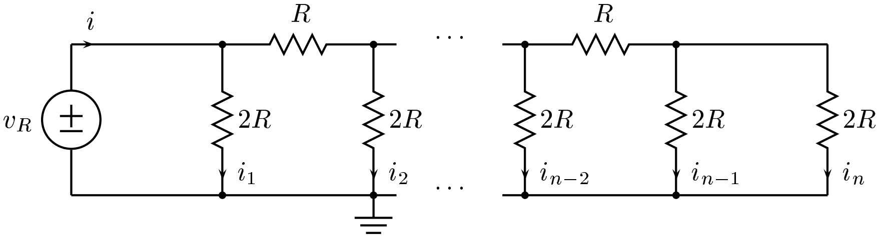
(b) Show that the output voltage can be expressed as

$$v_{out}(t) = -\frac{R_f}{R} v_R(b_1 2^{-1} + b_2 2^{-2} + \ldots + b_{n-1} 2^{-n+1} + b_n 2^{-n})$$



Solution:

(a) The voltage source see the ladder of resistors. Because the voltages at the two input terminals of the opamp are virtually equal to each other, all of the $2R$ resistors are essentially connected to ground. Thus from the point-of-view of the voltage source the circuit looks like



This is readily seen to be equivalent to a resistance of $R\Omega$. Thus, $i = v/R$. Using current dividers

$$i_1 = \frac{i}{2} = \frac{v}{2R}$$

$$
\begin{aligned}
i_2 &= \frac{i}{2^2} = \frac{v}{2^2 R} \\
&\vdots \\
i_n &= \frac{i}{2^n} = \frac{v}{2^n R}
\end{aligned}
$$

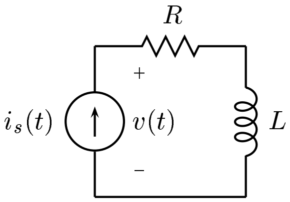
(b)

$$\frac{v_{out}(t)}{R_f} = b_1 i_1 + b_2 i_2 + \ldots + b_n i_n$$

or

$$v_{out}(t) = \frac{R_f v_R}{R}\left[b_1 2^{-1} + b_2 2^{-2} + \ldots + b_n 2^{-n}\right].$$

Problem 4.4: Find the value of the voltage $v(t)$ when the current $i_s(t)$ is the complex exponential time function $i_s(t) = e^{j\omega t}$.



Solution: The equivalent impedance is

$$Z_{eq}(j\omega) = R + j\omega L$$

Therefore,

$$
\begin{aligned}
V &= Z_{eq}(j\omega) I \\
&= (R + j\omega L) I.
\end{aligned}
$$

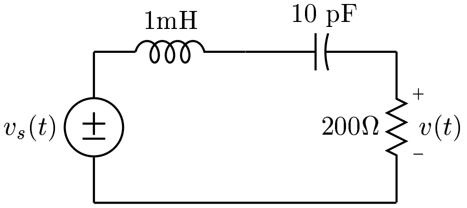
Since $I = 1$,

$$V = R + j\omega L$$

or

$$v(t) = (R + j\omega L) e^{j\omega t}$$

Problem 4.5: For the circuit below find $v(t)$ when $v_s(t) = \cos(\omega t)$.



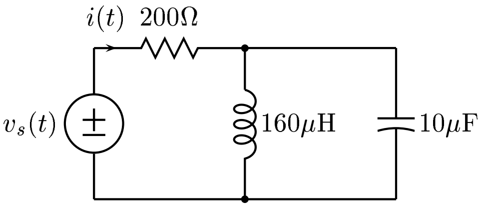
Solution: If the input were a complex exponential time function with a complex amplitude of V_s, then by using the voltage divider we would have

$$\begin{aligned} V &= V_s \frac{200}{200 + \frac{1}{j10^{-8}\omega} + j10^{-3}\omega} \\ &= V_s \frac{j(2\times 10^{10})\omega}{(1 - 10^5\omega^2) + j(2\times 10^{10})\omega} \end{aligned}$$

Since $v_s(t) = \cos(\omega t)$, we recognize that $V_s = 1$ and $v(t)$ is

$$\begin{aligned} v(t) &= \Re\mathrm{e}\left\{ \frac{j(2\times 10^{10})\omega}{1 - 10^5\omega^2 + j(2\times 10^{1}0)} e^{j\omega t} \right\} \\ &= \frac{2\times 10^{10}\omega}{\sqrt{(1-10^5\omega^2)^2 + (2\times 10^{10}\omega)^2}} \cos(\omega t + \frac{\pi}{2} - \left(\tan^{-1}\frac{2\times 10^{10}\omega}{1-10^5\omega^2}\right)) \\ &= \frac{2\times 10^{10}\omega}{\sqrt{(1-10^5\omega^2)^2 + (2\times 10^{10}\omega)^2}} \sin(\omega t - \left(\tan^{-1}\frac{2\times 10^{10}\omega}{1-10^5\omega^2}\right)) \end{aligned}$$

Problem 4.6: For the following circuit find $i(t)$ when $v_s(t) = \sin(\omega t)$.



Solution: For this circuit $i(t) = \Im\mathrm{m}(Y_{eq}e^{j\omega t})$. We can find the equivalent admittance by first finding the equivalent impedance.

$$Z_{eq} = R + \frac{jL\omega}{1 - LC\omega^2} = \frac{R - RLC\omega^2 + jL\omega}{1 - LC\omega^2}$$

Plugging in the component values, this is equal to

$$Z_{eq} = \frac{(3.2\times 10^{-7})\omega^2 + 200 + j(1.6\times 10^{-4})\omega}{1 - (1.6\times 10^{-9})\omega^2}$$

Thus

$$Y_{eq} = \frac{1-(1.6\times 10^{-9})\omega^2}{-(3.2\times 10^{-7})\omega^2 + 200 + j(1.6\times 10^{-4})\omega}$$

and

$$i(t) = |Y_{eq}|\sin(\omega t - \tan^{-1}\left(\frac{(1.6\times 10^{-4})\omega}{200-(3.2\times 10^{-7})\omega^2}\right)$$

The magnitude of the admittance is given by

$$|Y_{eq}| = \frac{|1-(1.6\times 10^{-9})\omega^2|}{(200-(3.2\times 10^{-7})\omega^2)^2 + (1.6\times 10^{-4}\omega)^2}$$
