Basis Pursuit with Sequential Measurements and Time-Varying Signals

(Dynamic updating for L1 minimization)

Salman Asif and **Justin Romberg**

School of Electrical and Computer Engineering

Georgia Institute of Technology

CAMSAP, Aruba Dec. 15, 2009

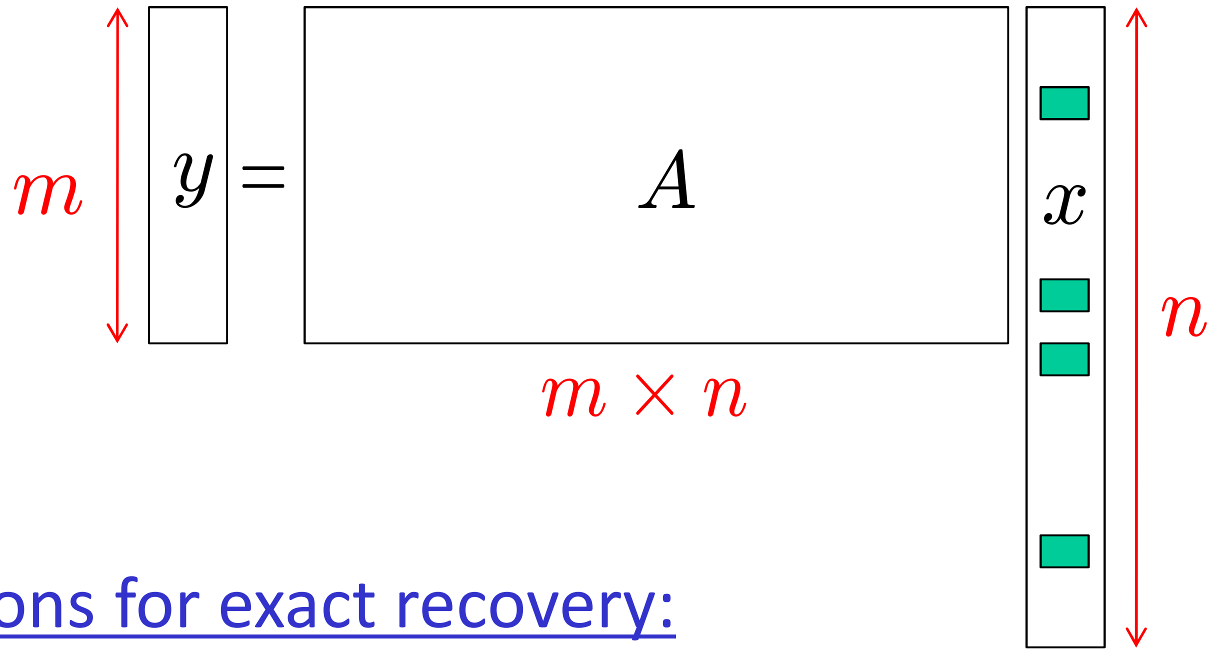
Outline

- Sparse signal recovery (Compressive sensing)
  - Basics (structure and randomness)
  - Reconstruction algorithms
  - A look at least squares
- Dynamic updating
  - Least squares update (RLS)
  - L1 update (BP, BPDN, DS)
  - Homotopy schemes

Introduction (CS)

- Compressive Sensing (CS) (or sparse approximation)

$$\boxed{y = Ax}$$



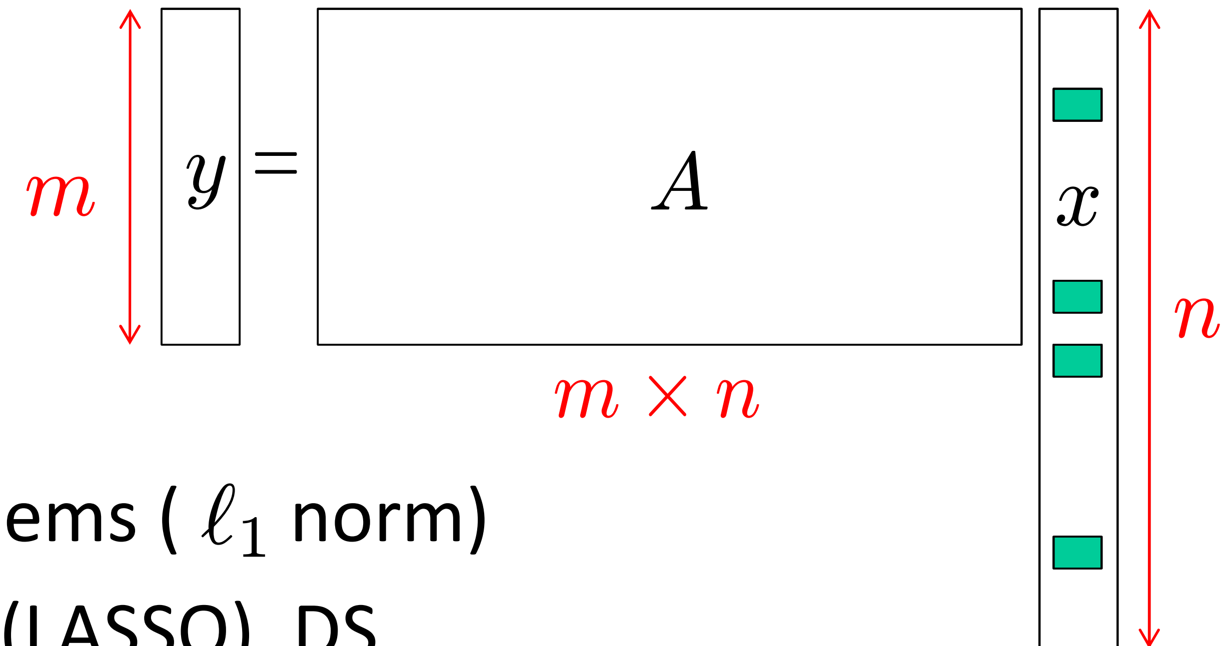
Typical assumptions for exact recovery:

- x is *sparse:* Very few nonzero elements
- A is *incoherent*: Columns are dissimilar

Introduction (CS)

- Compressive Sensing (CS) (or sparse approximation)

$$\boxed{y = Ax}$$

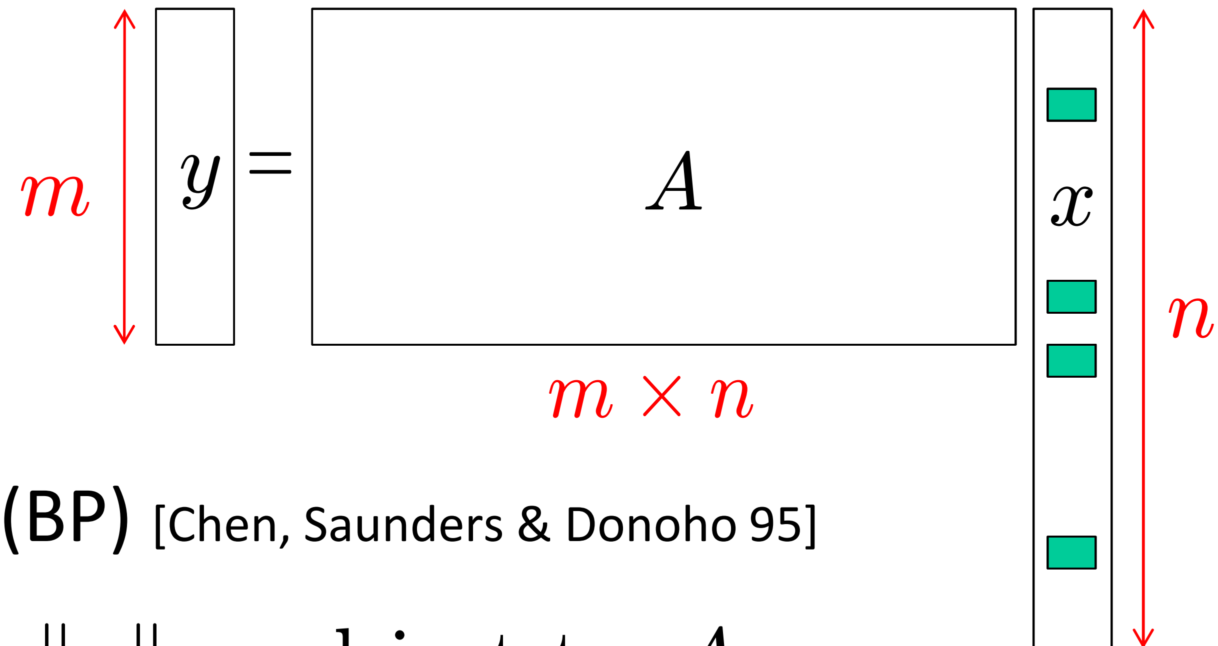


- Convex problems (ℓ_1 norm)
  - BP, BPDN (LASSO), DS
- Greedy approximiation
  - OMP, StOMP, ROMP, CoSaMP, …

Introduction (CS)

- Compressive Sensing (CS)

$$\boxed{y = Ax}$$



- Basis Pursuit (BP) [Chen, Saunders & Donoho 95]

$$\text{minimize } \|x\|_1 \ \text{ subject to } \ Ax = y$$

- Lots of guarantees for exact recovery!

[Donoho; Candes, Romberg & Tao, …]

Introduction (CS)

- Compressive Sensing (CS)

$$y = Ax + e$$

When measurements are *noisy* and/or x is *nearly* sparse (*compressible*)

- Basis Pursuit Denoising (BPDN) [Chen, Saunders & Donoho 95]

$$\text{minimize } \tau\|x\|_1 + \frac{1}{2}\|Ax - y\|_2^2$$

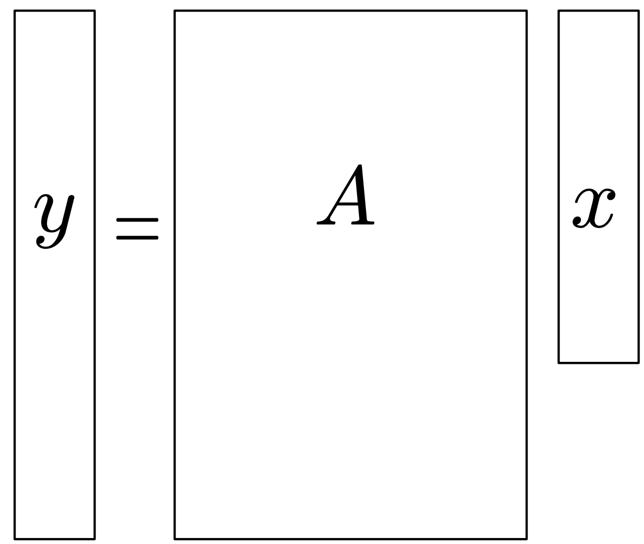
- Dantzig Selector (DS) [Candes & Tao 05]

$$\text{minimize } \|x\|_1 \text{ subject to } \|A^T(Ax - y)\|_\infty \leq \tau$$

Introduction (LS)

- Least Squares (LS) Model

$$y = Ax$$



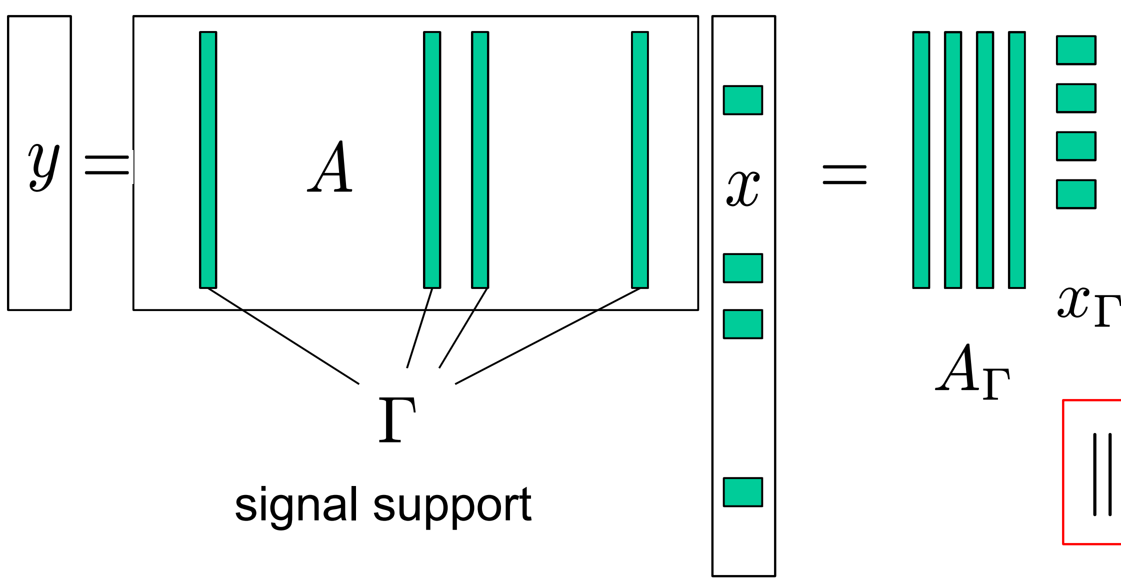
  - A is *full rank*
  - x is *arbitrary*

- LS estimate

$$\text{minimize } \|Ax - y\|_2 \quad \rightarrow \quad x_0 = (A^T A)^{-1} A^T y$$

Diversion (LS & CS)

LS (over-determined)	**CS** (under-determined)
Unknown signal is arbitrary	Unknown signal has sparse structure
A has full column rank	A is nearly isometric on the signal support
Closed form solution exists	Solve a convex program



"RIP"

$$\boxed{\|A_\Gamma x_\Gamma\|_2 \approx \|x_\Gamma\|_2}$$

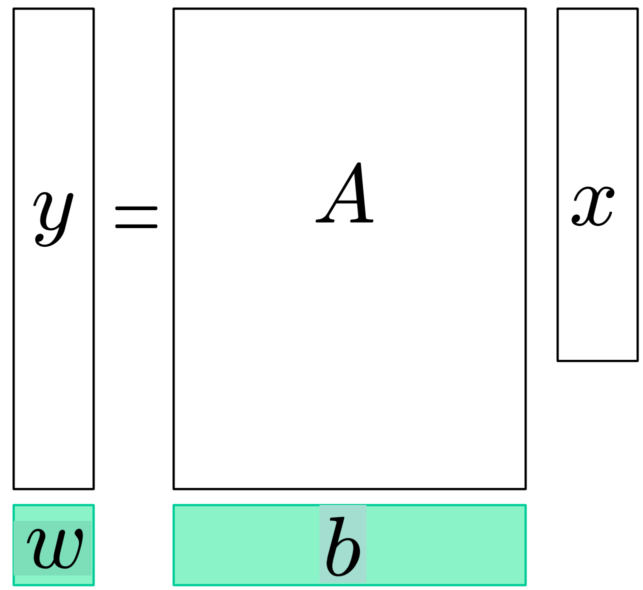
LS & CS

- Solution x_o (if supports and signs (Γ,z) are known)
  - LS (A full rank) solution:
$$x_0 = (A^T A)^{-1} A^T y$$
  - BP solution:
$$(x_0)_\Gamma = (A_\Gamma^T A_\Gamma)^{-1} A_\Gamma^T y$$
  - BPDN solution:
$$(x_0)_\Gamma = (A_\Gamma^T A_\Gamma)^{-1} (A_\Gamma^T y - \tau z)$$
  - DS solution
$$(x_0)_\Gamma = (A_{\Gamma_d}^T A_\Gamma)^{-1} (A_{\Gamma_d}^T y + \tau z_d)$$

Motivation for Dynamic Update (RLS)

- Measurement Update Model

$$\begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} A \\ b \end{bmatrix} x$$



- Recursive LS

$$x_1 = (A^T A + b^T b)^{-1}(A^T y + b^T w)$$
$$= x_0 + K_1(w - bx_0)$$

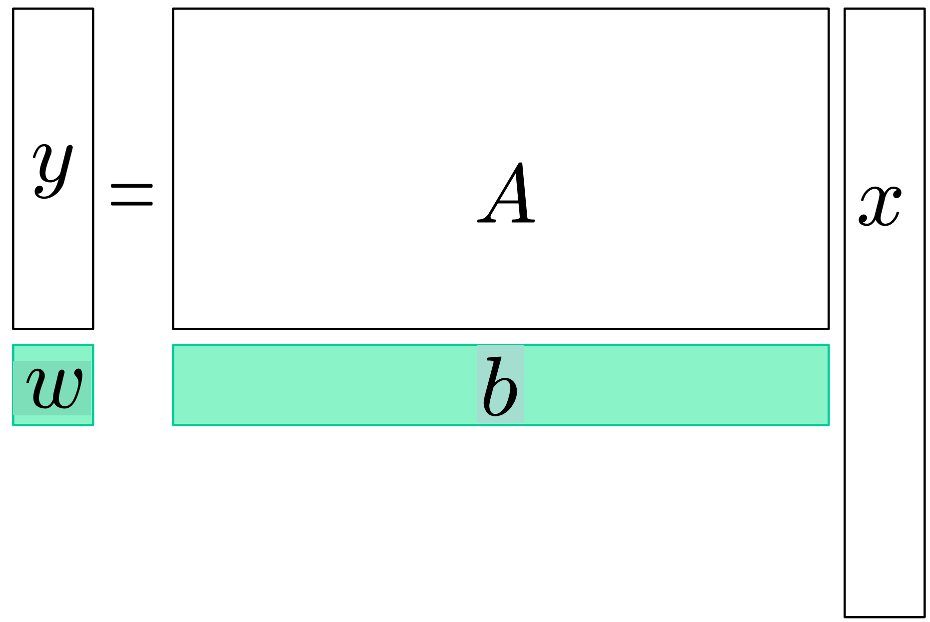
Rank one update

$$K_1 = (A^T A)^{-1} b^T (1 + b(A^T A)^{-1} b^T)$$

Measurement Update (CS)

- Update Model

$$\begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} A \\ b \end{bmatrix} x$$



- BP update

$$\text{minimize } \|x\|_1 \;\; \text{s.t.} \;\; Ax = y, \;\; \underline{bx = w}$$

Additional constraint for new measurement

Update the solution (quickly)!?!?

Measurement Update (CS)

- Update Model

$$\begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} A \\ b \end{bmatrix} x + \begin{bmatrix} e_y \\ e_w \end{bmatrix}$$

- Basis Pursuit Denoising (BPDN)

$$\text{minimize } \tau \|x\|_1 + \frac{1}{2}\left[\|Ax - y\|_2^2 + \|bx - w\|_2^2\right]$$

- Dantzig Selector (DS)

$$\text{minimize } \|x\|_1 \ \text{ s.t. } \ \|A^T(Ax - y) + b^T(bx - w)\|_\infty \leq \tau$$

Homotopy (Sequential measurements)

- "Transform the original optimization problem into an easy (but related) form. Traverse the homotopy path towards the original solution."
- BP:

$$\text{minimize } \|x\|_1 \ \text{ s.t. } \ Ax = y, (1-\epsilon)bx_0 + \epsilon\, bx = w$$

- BPDN:

$$\text{minimize } \tau\|x\|_1 + \frac{1}{2}\left[\|Ax - y\|_2^2 + \epsilon\, \|bx - w\|_2^2\right]$$

- DS:

$$\text{minimize } \|x\|_1 \ \text{ s.t. } \ \|A^T(Ax - y) + \epsilon\, b^T(bx - w)\|_\infty \leq \tau$$

Piecewise linear homotopy paths; it takes a series of rank-one updates, as ϵ is changed 0 → 1

BPDN Homotopy Update

- Optimality conditions for solution x_0 at given τ

$$\text{minimize } \tau\|x\|_1 + \frac{1}{2}\|Ax - y\|_2^2$$

$$\partial_x(\tau\|x\|_1 + \tfrac{1}{2}\|Ax - y\|_2^2) \;\rightarrow\; \|A^T(Ax - y)\|_\infty \leq \tau$$

$$\boxed{\begin{array}{l} 1.\;\; A_\Gamma^T(Ax_0 - y) = -\tau z \\[2ex] 2.\;\; \|A_{\Gamma^c}^T(Ax_0 - y)\|_\infty < \tau \end{array}}$$

BPDN Homotopy Update

- Optimality conditions for solution x_0 at given τ

$$\text{minimize } \tau\|x\|_1 + \frac{1}{2}\|Ax - y\|_2^2$$

1. $A_\Gamma^T(Ax_0 - y) = -\tau z$
2. $\|A_{\Gamma^c}^T(Ax_0 - y)\|_\infty < \tau$

- Fix τ, construct the homotopy path with ϵ for

$$\text{minimize } \tau\|x\|_1 + \tfrac{1}{2}\left[\|Ax - y\|_2^2 + \epsilon\,\|bx - w\|_2^2\right]$$

- Iteratively change ϵ from 0 to 1, while obeying the new optimality conditions.
- Each step involves a rank-one update

BPDN Homotopy Update

$$\text{minimize } \tau\|x\|_1 + \frac{1}{2}\left[\|Ax - y\|_2^2 + \epsilon\,\|bx - w\|_2^2\right]$$

- Optimality conditions for solution x_k at ϵ_k

$$\boxed{\begin{array}{l} 1.\;\; A_\Gamma^T(Ax_k - y) + \epsilon_k b_\Gamma^T(bx_k - w) = -\tau z \\ 2.\;\; \|A_{\Gamma^c}^T(Ax_k - y) + \epsilon_k b_{\Gamma^c}^T(bx_k - w)\|_\infty < \tau \end{array}}$$

- Direction of x with increase in ϵ: $\epsilon_k \rightarrow \epsilon_k^+$

$$\partial x = \begin{cases} -(\epsilon_k^+ - \epsilon_k)(A_\Gamma^T A_\Gamma + \epsilon_k^+ b_\Gamma^T b_\Gamma)^{-1} b_\Gamma^T(bx_k - w) & \text{on } \Gamma \\ 0 & \text{otherwise,} \end{cases}$$

- Increase ϵ until there is one element change in the support. Repeat until ϵ = 1.

BP Homotopy Update

- BP (Primal): $\min. \; \|x\|_1 \;\text{ s.t. }\; Ax = y$
- BP (Dual): $\max. \; -\lambda^T y \;\text{ s.t. }\; \|A^T\lambda\|_\infty \leq 1$
- Opt. conditions for primal-dual pair (x_o,λ_o):

$$Ax_0 = y, \quad A_\Gamma^T \lambda_0 = -z, \quad \|A_{\Gamma^c}^T \lambda_0\|_\infty < 1,$$

- Construct the homotopy for the primal-dual pair

$$\min. \; \|x\|_1 \;\text{ s.t. }\; Ax = y, (1-\epsilon)bx_0 + \epsilon\, bx = w$$

$$\max. \; -\lambda^T y - \nu^T[(1-\epsilon)bx_0 + \epsilon w] \;\text{ s.t. }\; \|A^T\lambda + b^T\nu\|_\infty \leq 1$$

- Increase ϵ from 0 to 1, while obeying opt. conditions

BP Homotopy Update

$$\min.\ \|x\|_1 \ \text{ s.t. } \ Ax = y, (1-\epsilon)bx_0 + \epsilon\, bx = w$$

$$\max.\ -\lambda^T y - \nu^T[(1-\epsilon)bx_0 + \epsilon w] \ \text{ s.t. } \ \|A^T\lambda + b^T\nu\|_\infty \leq 1$$

- Optimality conditions for solution at ϵ_k (x_k,λ_k,ν_k):

$$\boxed{\begin{array}{c} Ax_k = y, \quad bx_k = (1-\epsilon_k)bx_0 + \epsilon_k w, \\ A_\Gamma^T\lambda_k + b_\Gamma^T\nu_k = -z, \quad \|A_{\Gamma^c}^T\lambda_k + b_{\Gamma^c}^T\nu_k\|_\infty < 1 \end{array}}$$

- Direction of x with increase in ϵ: $\epsilon_k \rightarrow \epsilon_k^+$

$$\partial x = \begin{cases} (\epsilon_k^+ - \epsilon_k)\begin{bmatrix} A_\Gamma \\ b_\Gamma \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0}_m \\ -bx_0 + w \end{bmatrix} & \text{on } \Gamma \\ 0 & \text{otherwise} \end{cases}$$

- Increase ϵ until one element in x shrinks to zero.
- Then update dual vector to find new support!

Time-Varying Sparse Signals

- Back to the original system: $y = Ax + e$
- Sparse signal changes slightly and we get a new set of measurements (A is fixed)

$$x \to \breve{x}; \qquad y \to \breve{y} = A\breve{x} + \breve{e}$$

- BP: $\min.\ \|x\|_1 \ \text{ s.t. } \ Ax = \breve{y}$
- BPDN: $\min.\ \tau\|x\|_1 + \frac{1}{2}\|Ax - \breve{y}\|_2^2$
- DS: $\min.\ \|x\|_1 \ \text{ s.t. } \ \|A^T(Ax - \breve{y})\|_\infty \leq \tau$

Homotopy (Time-varying signals)

- Back to the original system: $y = Ax + e$
- Sparse signal changes slightly and we get a new set of measurements (A is fixed)

$$x \to \breve{x}; \qquad y \to \breve{y} = A\breve{x} + \breve{e}$$

- BP: $\min.\ \|x\|_1 \ \text{s.t.}\ Ax = (1-\epsilon)y + \epsilon\breve{y}$
- BPDN: $\min.\ \tau\|x\|_1 + \frac{1}{2}\|Ax - (1-\epsilon)y - \epsilon\breve{y}\|_2^2$
- DS: $\min.\ \|x\|_1 \ \text{s.t.}\ \|A^T(Ax - (1-\epsilon)y - \epsilon\breve{y})\|_\infty \leq \tau$

Summary

- Dynamic updating for L1 minimization
- Instead of solving new optimization problems from scratch, use homotopy!
- Each homotopy step uses rank-one update
- Small number of homotopy steps to update!

  (mainly it depends on the problem parameters)

Thankyou !

Salman Asif Justin Romberg

{sasif, jrom}@gatech.edu

Papers:

- *Dynamic updating for L1 minimization, To appear in IEEE JSTSP issue on CS*
- *Basis pursuit with sequential measurements and …, CAMSAP, Aruba*

Matlab: *http://users.ece.gatech.edu/~sasif/homotopy*

Questions