Channel Protection: Random Channel Meets Sparse Channels

[a “blind deconvolution” scheme]

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joint work with **William Mantzel** *and* **Justin Romberg**

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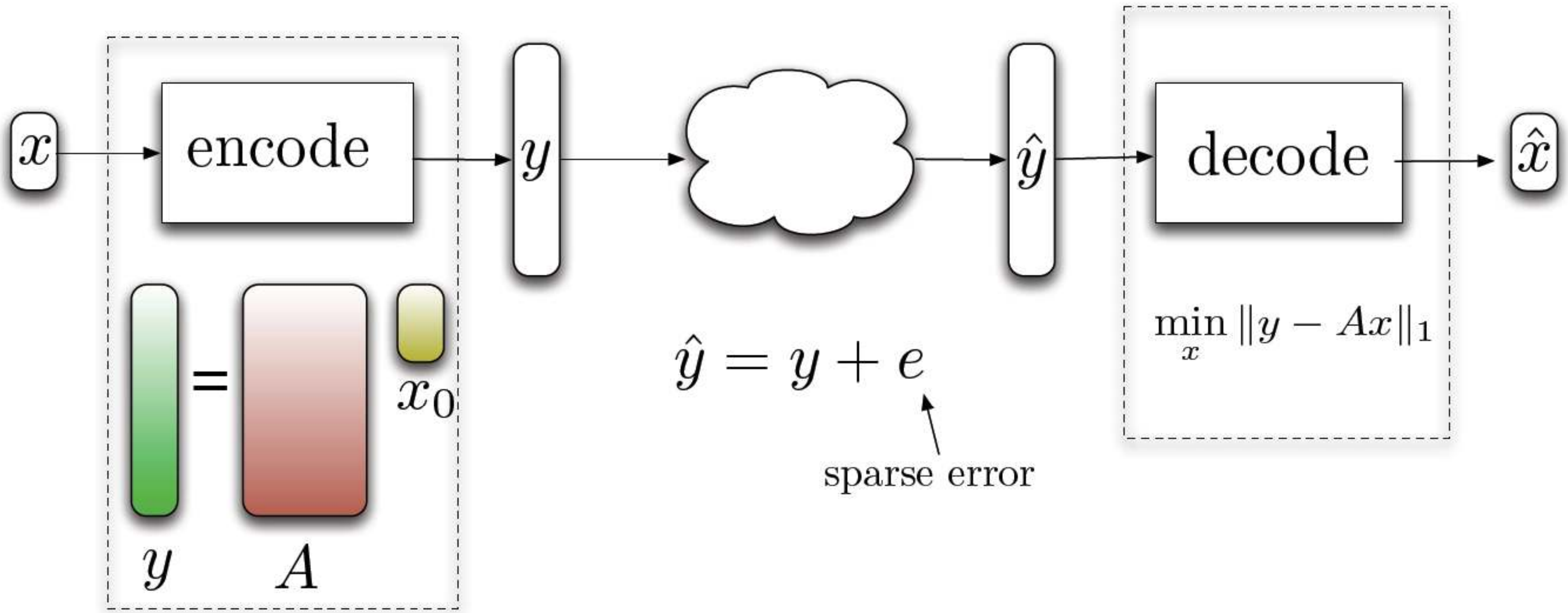
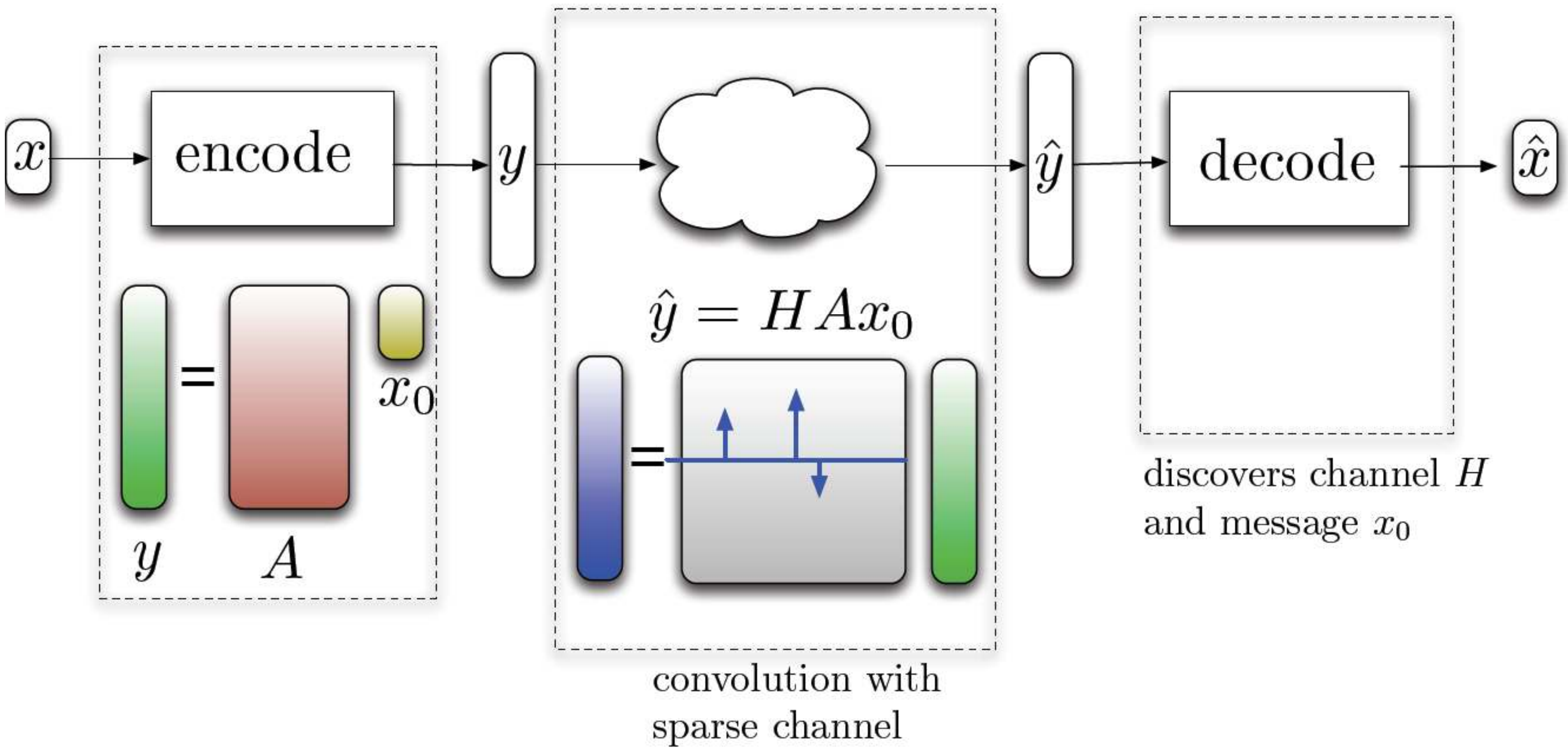
Random Coding Example [Candes & Tao '04]



- Structure: A is a *tall* random (mxn) matrix, e is sparse
- It can recover the signal x exactly if

  |supp(e)| < Const . (m-n)/log(m)
- Also stable against small disturbances [Candes & Randall '07]

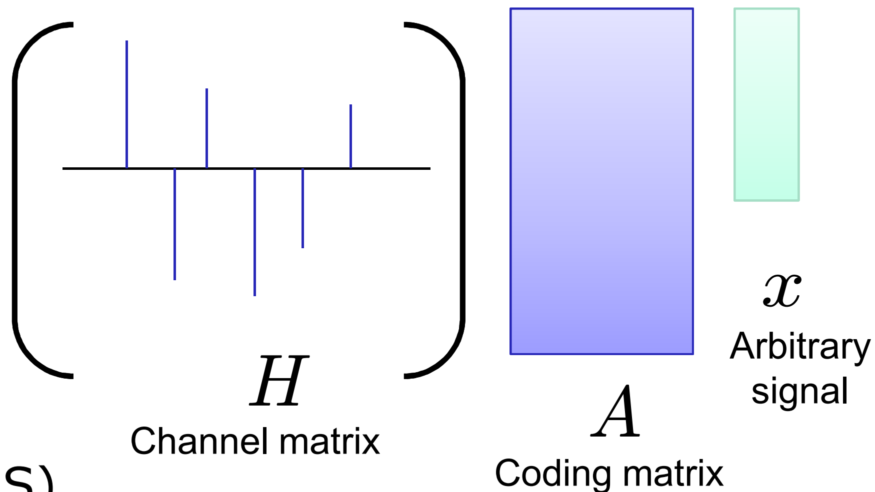
Random Coding and Blind Deconvolution



Problem Formulation

- Problem setup:

$$y = h * Ax \longrightarrow$$



- x – unknown message
- A – m x n matrix
- h – channel impulse response
  (length of channel L; sparsity S)

- Message is encoded with a tall random matrix before transmission: Ax
- Signal received via a channel with impulse response h

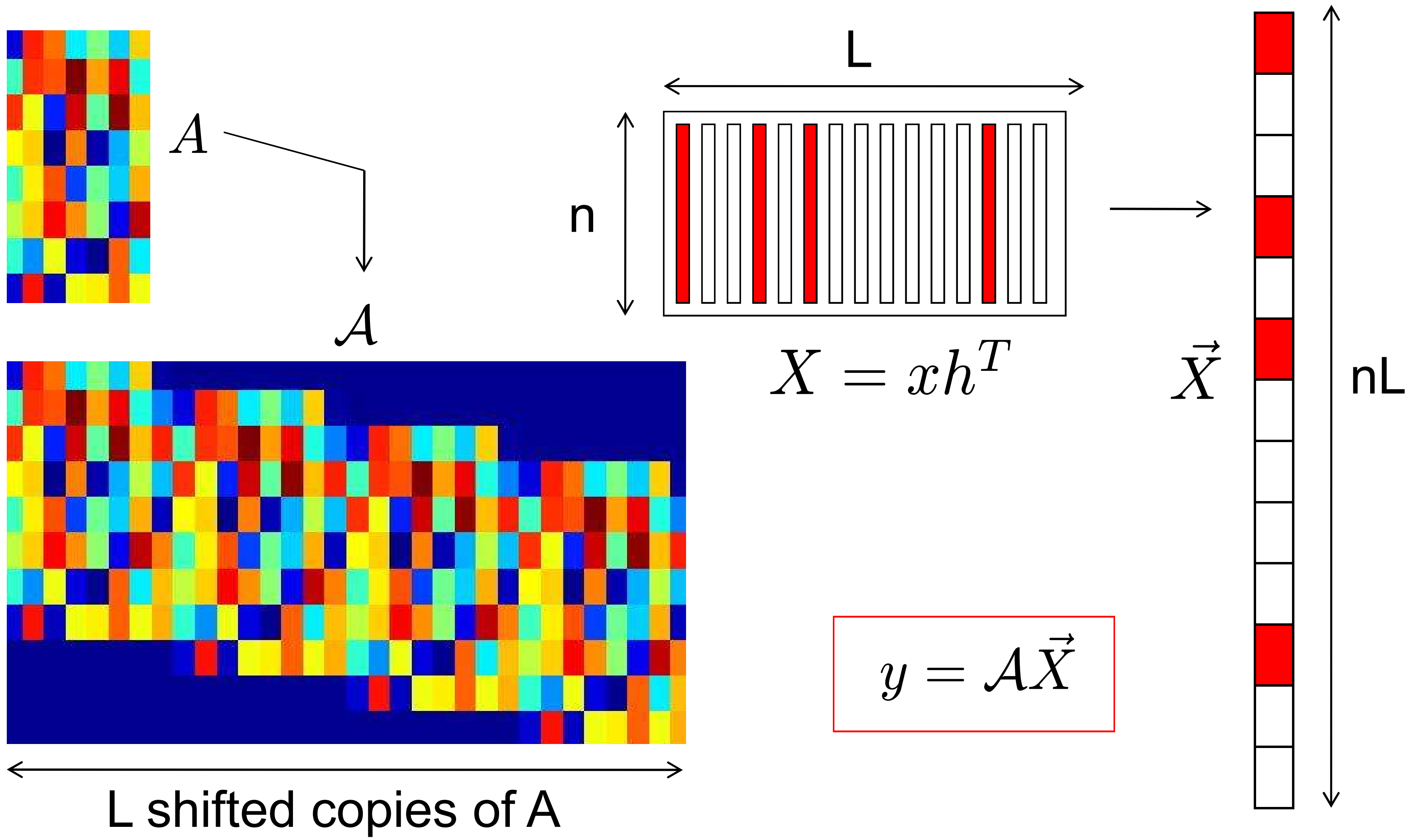
- Goal: Estimate x without knowledge of h

Random Coding for Blind Deconvolution, why?

- "Channel protection": $y = h * Ax$
- Random coding may provide robustness against (*sparse) convolutive"* channels (now we have a structure, aha!)
- Structure: Transmitted signal lies in a *smaller subspace*: range(A), channel impulse response is *sparse*
- The recovery performance improves as we increase redundancy
- Goal: Estimate x without knowledge of h with minimum redundancy in the coding
- The problem of estimating x and h simultaneously can be nonconvex

Deconvolution

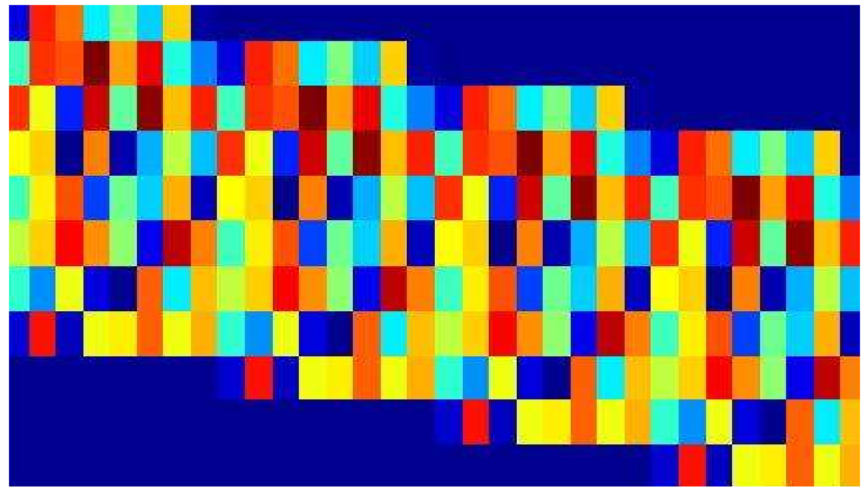
- Problem variables can be described as rank one matrix

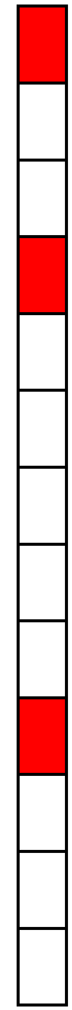


Proposed Methods for Deconvolution

- Solve the following inverse problem

$$\boxed{y = \mathcal{A}\vec{X}}$$

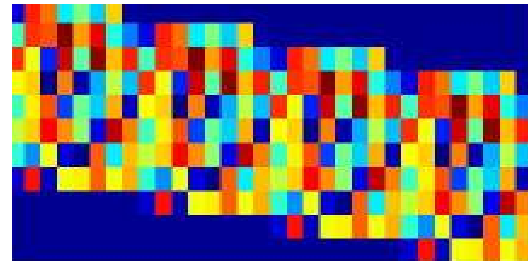




- **Block L1**
- **Rank constrained block L1**
- **Alternating minimization**
- **Rank minimization**

Block L1

- Naïve approach



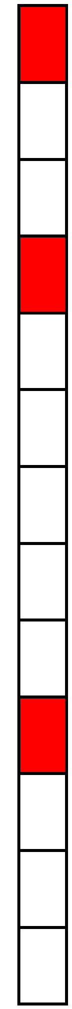
$$\text{minimize } \|\vec{X}\|_1 \text{ subject to } y = \mathcal{A}\vec{X}$$

- However, sparsity pattern is not arbitrary *(block sparse)*

- The block-L1 approach improves performance (slightly)
  [Baraniuk et al., Stojnic et al., Blumensath et al. 2008]

$$\text{minimize } \sum_k \|\vec{X}\|_2 \text{ subject to } y = \mathcal{A}\vec{X}$$

- This requires m = O(nS log(nL)) for perfect recovery
- However, there are only (n+S) degrees of freedom!
- We want to break through to get m = O(n+S)
  (modulo log factor)



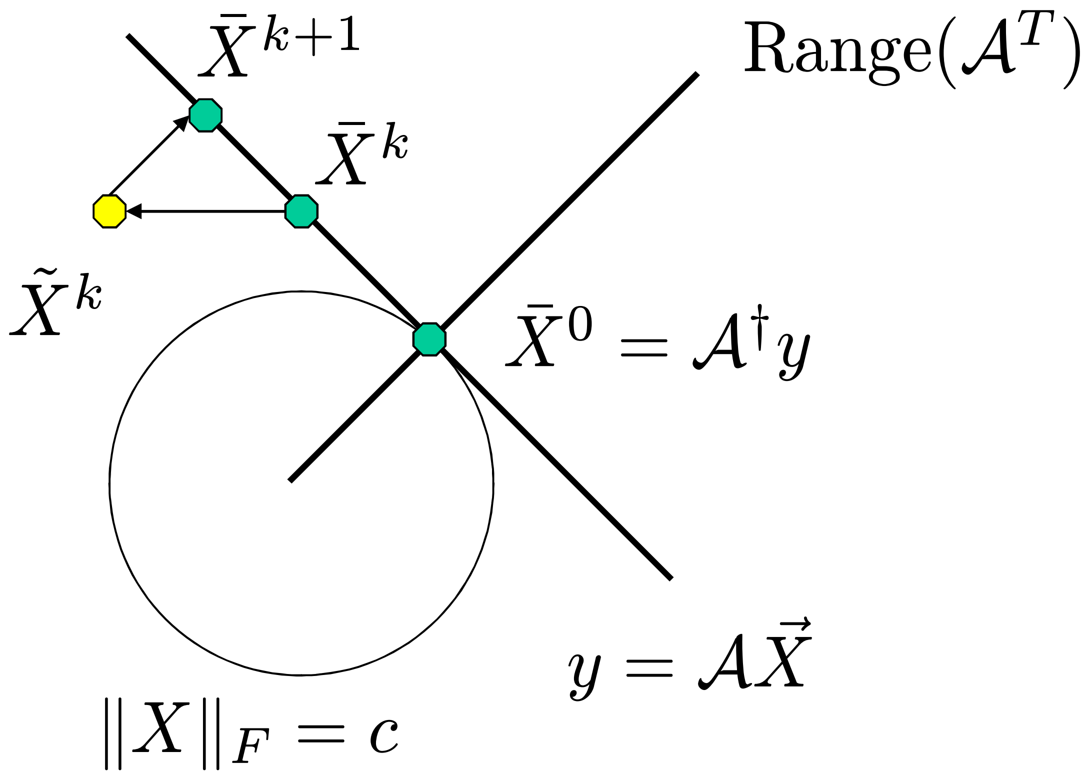
Rank Constrained Block-L1

- Enforce the rank-1 constraint on block L1 (nonconvex)
- Gradient descent with rank-constraint and affine projection
- Traverse the affine subspace of measurement constraints
  - Project X to the nearest rank-1 matrix
  - Minimize 2→1 norm: $\sum \|X_k\|_2$
  - Project back to the affine subspace
- When $X = f\,g^T$ (rank 1), this minimizes $\|f\|_2 \|g\|_1$
- Enforces sparsity on the number of active columns
- Rank-1 projection at every step
- Gradient descent: The update direction for each nonzero column:

$$\Delta X_k = -\frac{X_k}{\|X_k\|_2}$$

Rank Constrained Block-L1

- Find closest rank one matrix to the solution (SVD)
- Minimize 2→1 norm (can be combined with SVD as soft thresholding)
- Project onto the affine space defined by the measurements



$$U\Sigma V^T = \bar{X}^k$$

$$\tilde{X}^k = \sigma_1 u_1 \tilde{v}_1^T$$

$$\bar{X}^{k+1} = P_{\mathcal{A}\vec{X}=y}\tilde{X}^k$$

Soft threshold:

$$\tilde{v}_1 = S_\epsilon(v_1)$$

Alternating Minimization (AM)

- Solve the joint optimization problem

$$\text{minimize } \frac{1}{2}\|h * Ax - y\|_2^2 + \tau\|h\|_1$$

- Hard to solve simultaneously
- Solve *alternately* for x and h
- BPDN for h (using previous estimate of x)

$$\text{minimize } \frac{1}{2}\|M_{Ax}h - y\|_2^2 + \tau\|h\|_1$$

- Least squares for x (using previous estimate of h)

$$\text{minimize } \|M_{h*A}x - y\|_2$$

- Iterate until convergence!

Discussion About Analysis of AM

- What do we know* about the setup

$$\text{minimize } \frac{1}{2}\|h * Ax - y\|_2^2 + \tau\|h\|_1$$

- What if we know the signal perfectly:

$$\text{minimize } \tfrac{1}{2}\|Gh - y\|_2^2 + \tau\|h\|_1$$

recovers the signal exactly if $m \geq S \log^5 L$

- What if we know the channel perfectly:

$$\text{minimize } \|HAx - y\|_2$$

for a generic h with S taps, matrix HA has rank n if

m > Const . (n+S) log (m)

This follows from quantitative uncertainty principle!

* Do not ask what happens if do not know x and h

Rank Minimization

- Find minimum rank matrix satisfying the measurements

$$\text{minimize } \operatorname{rank}(X) \;\; \text{subject to} \;\; \mathcal{A}\vec{X} = y$$

- Convex relaxation -- Nuclear norm minimization [Fazel 2002, Recht et al. 2007, …]

$$\text{minimize } \|X\|_* \;\; \text{subject to} \;\; \mathcal{A}\vec{X} = y$$

- Equivalent semidefinite program

$$\begin{aligned} \text{minimize} \;\; & \frac{1}{2}(\operatorname{Tr}(W_1) + \operatorname{Tr}(W_2)) \\ \text{subject to} \; & \begin{bmatrix} W_1 & X \\ X^T & W_2 \end{bmatrix} \succeq 0 \\ & \mathcal{A}\vec{X} = y \end{aligned}$$

Rank Minimization

- SDP formulation minimizes rank; indifferent to sparsity
- Equivalently minimizes $\|h\|_2^2 + \|x\|_2^2$
- Additional constraints:

Norm of h

$$\mathrm{Tr}(W_2) = 1$$

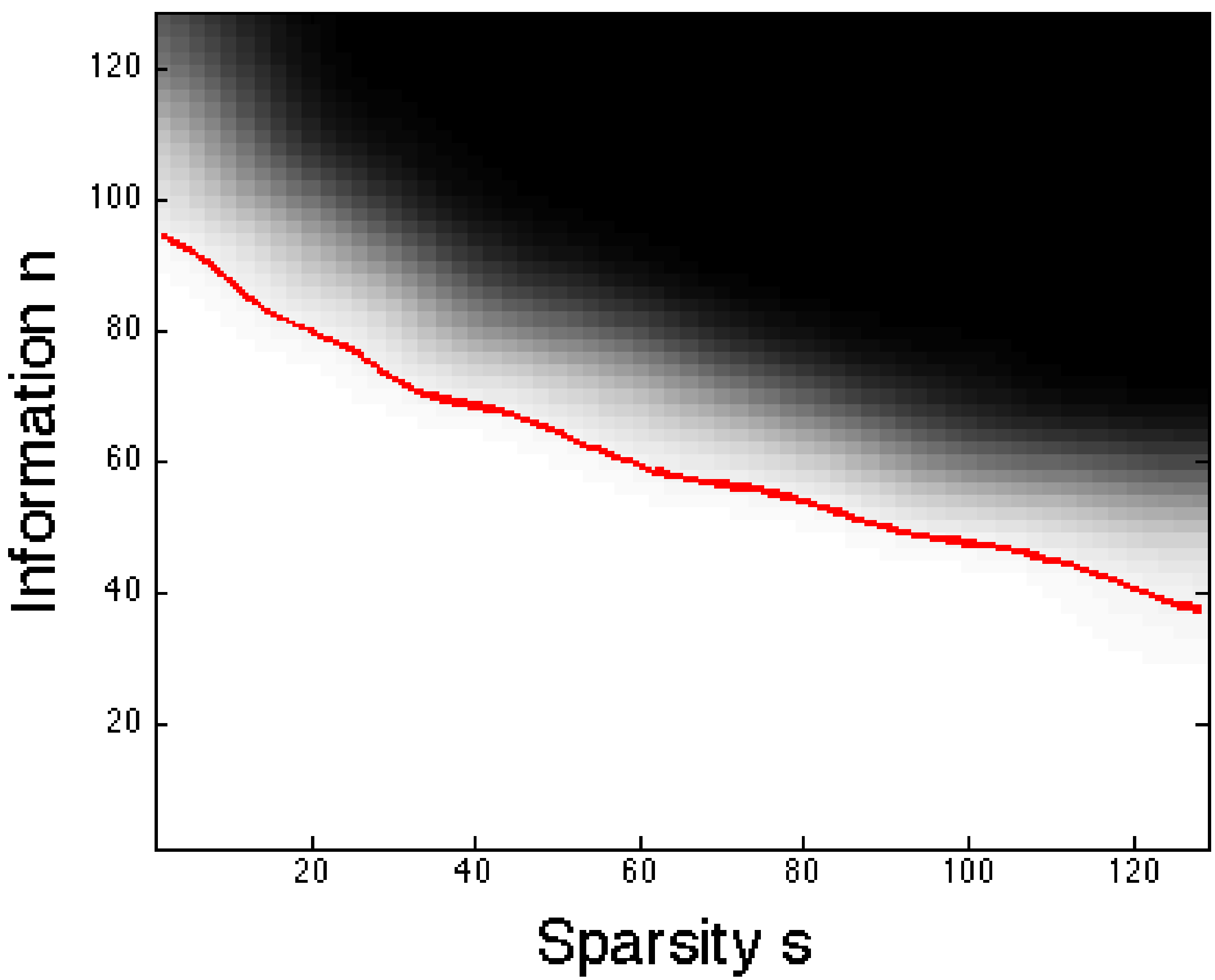
$$\text{i.e. } \|h\|_2 = 1$$

Sparsity of the matrix

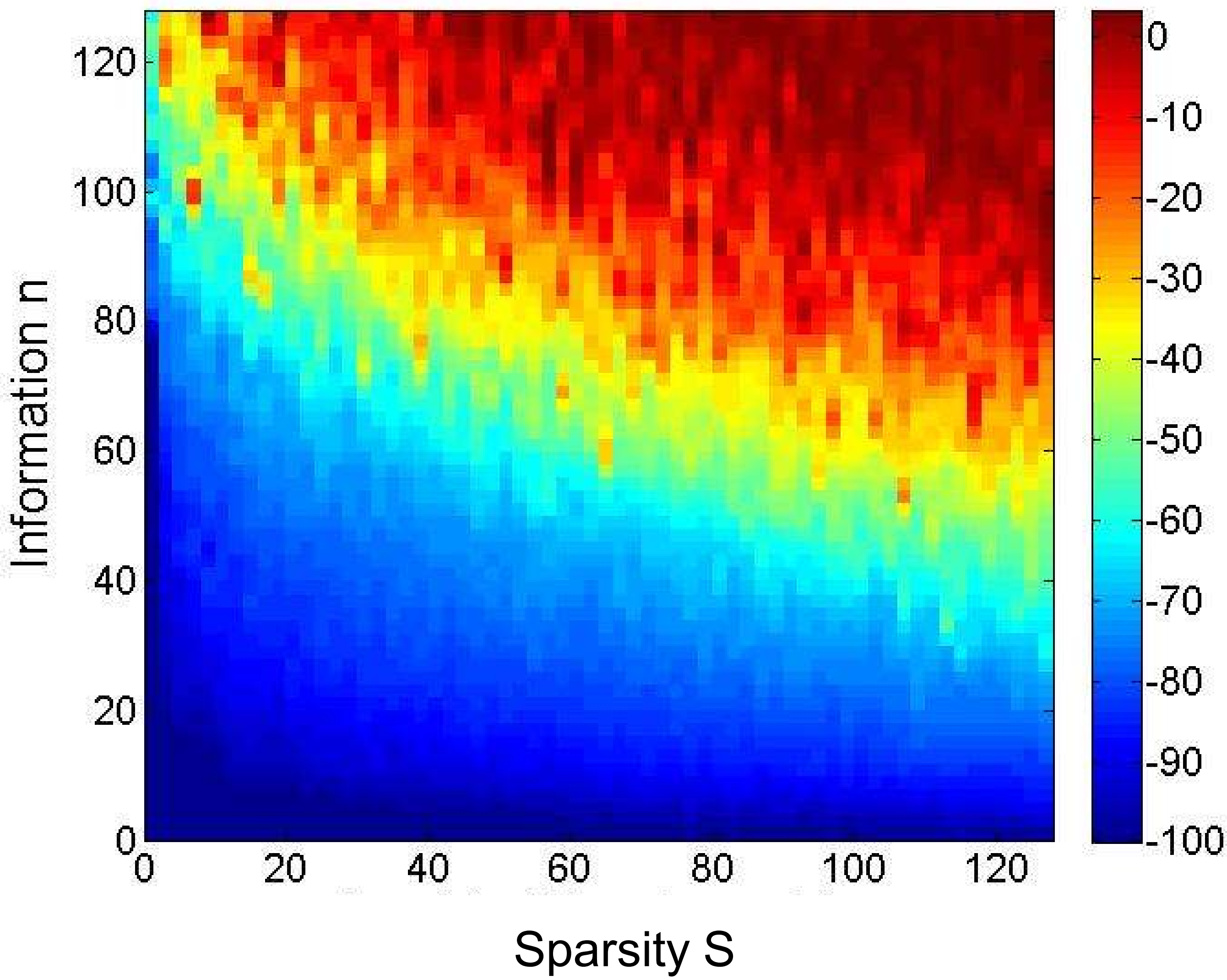
$$\mathbf{1}^T |W_2| \mathbf{1} \leq k$$

$$\text{i.e. } \|h\|_2 \leq k$$

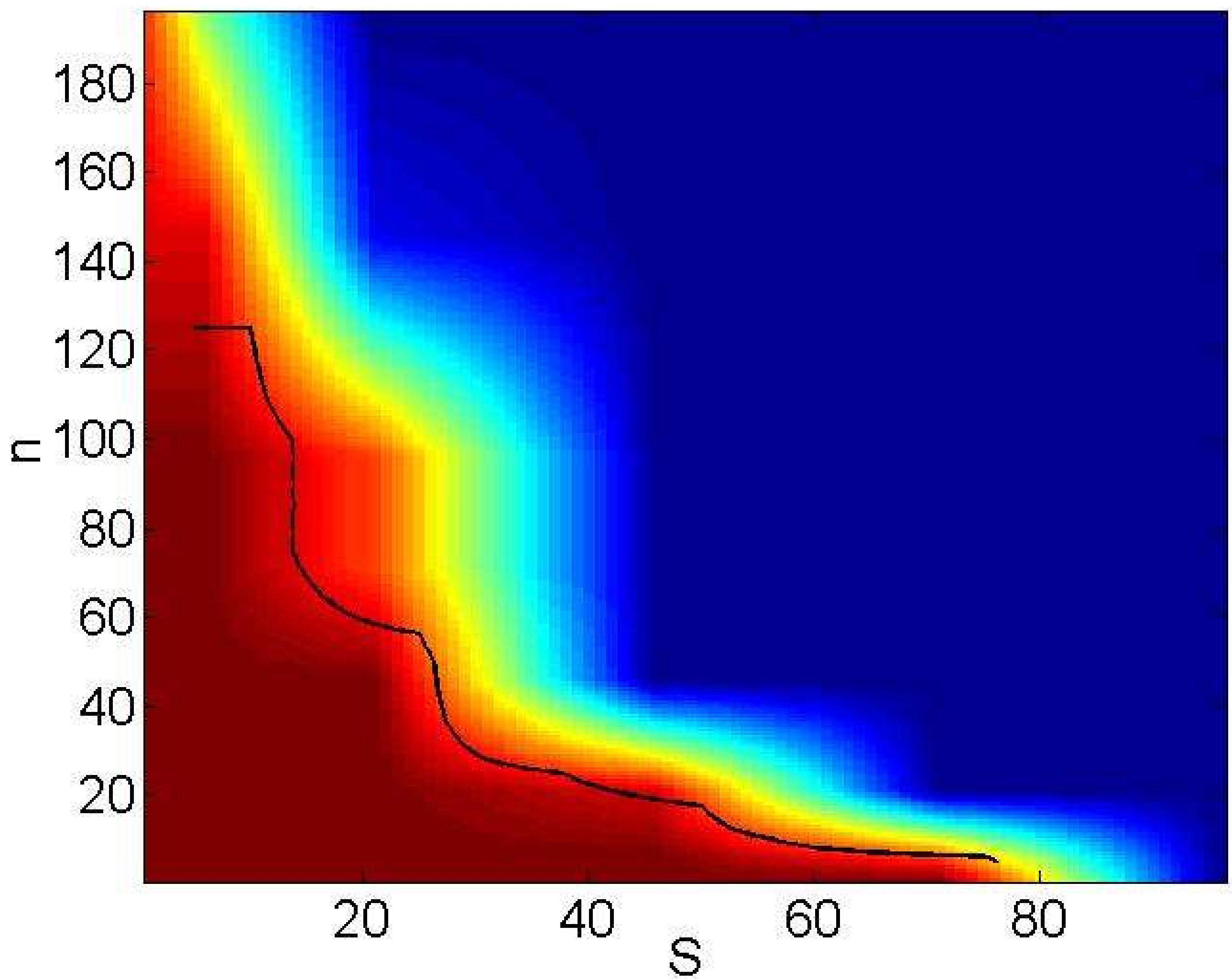
Results: Rank constrained Gradient Descent (m=256, L=128)



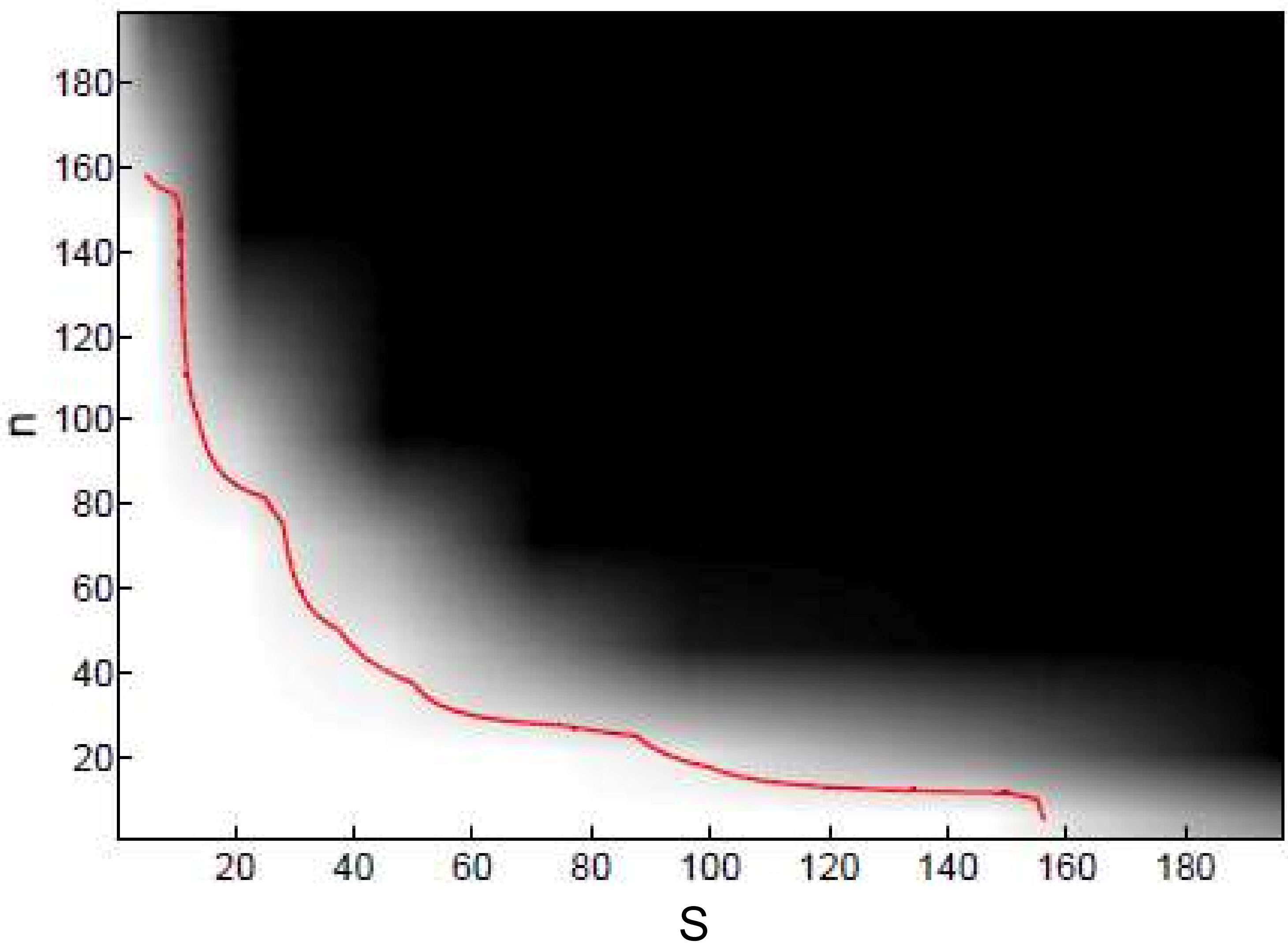
Results: Rank constrained Gradient Descent (m=256, L=128)



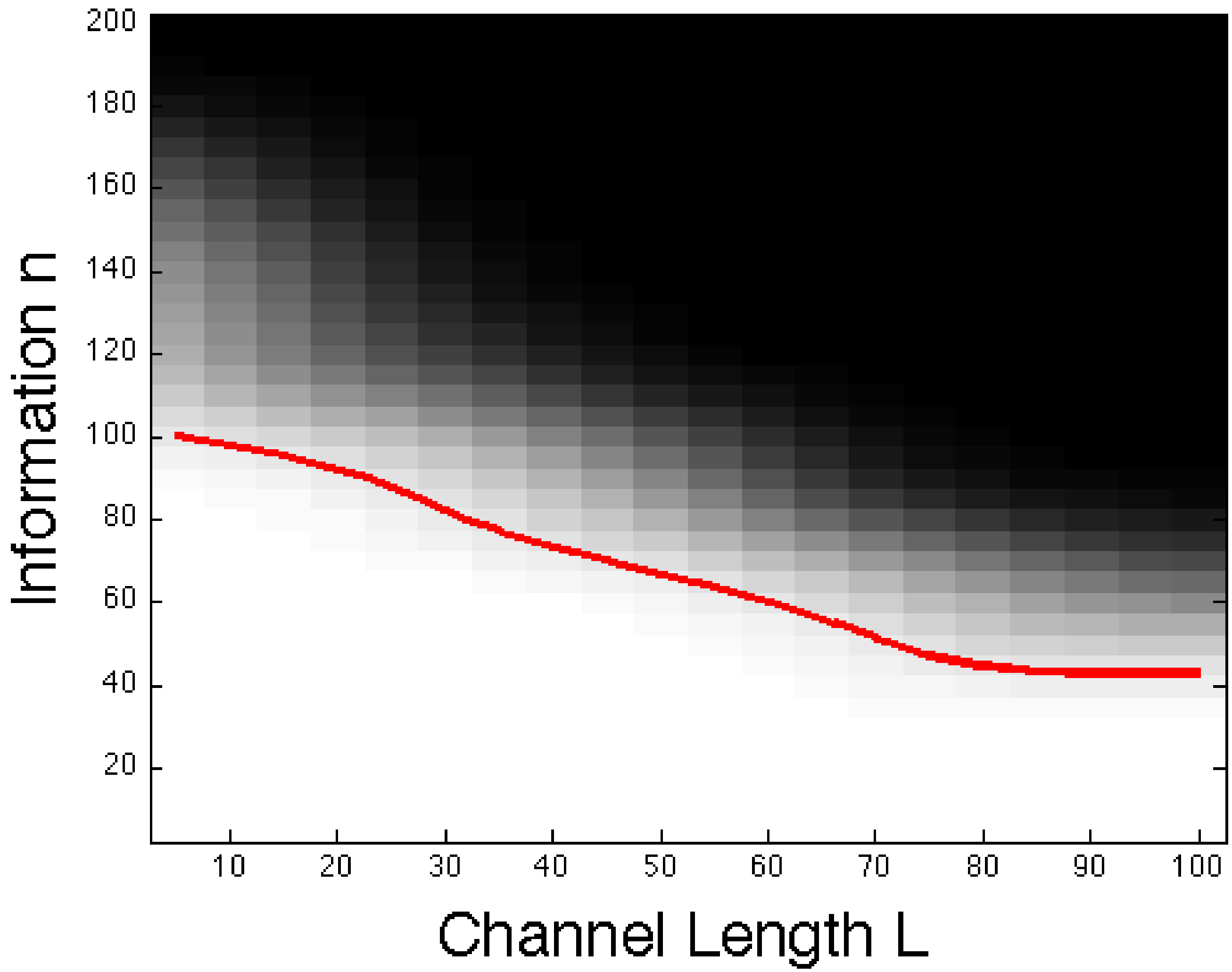
Results: Alternating minimization (m=256, L=128, Linear convolution model)



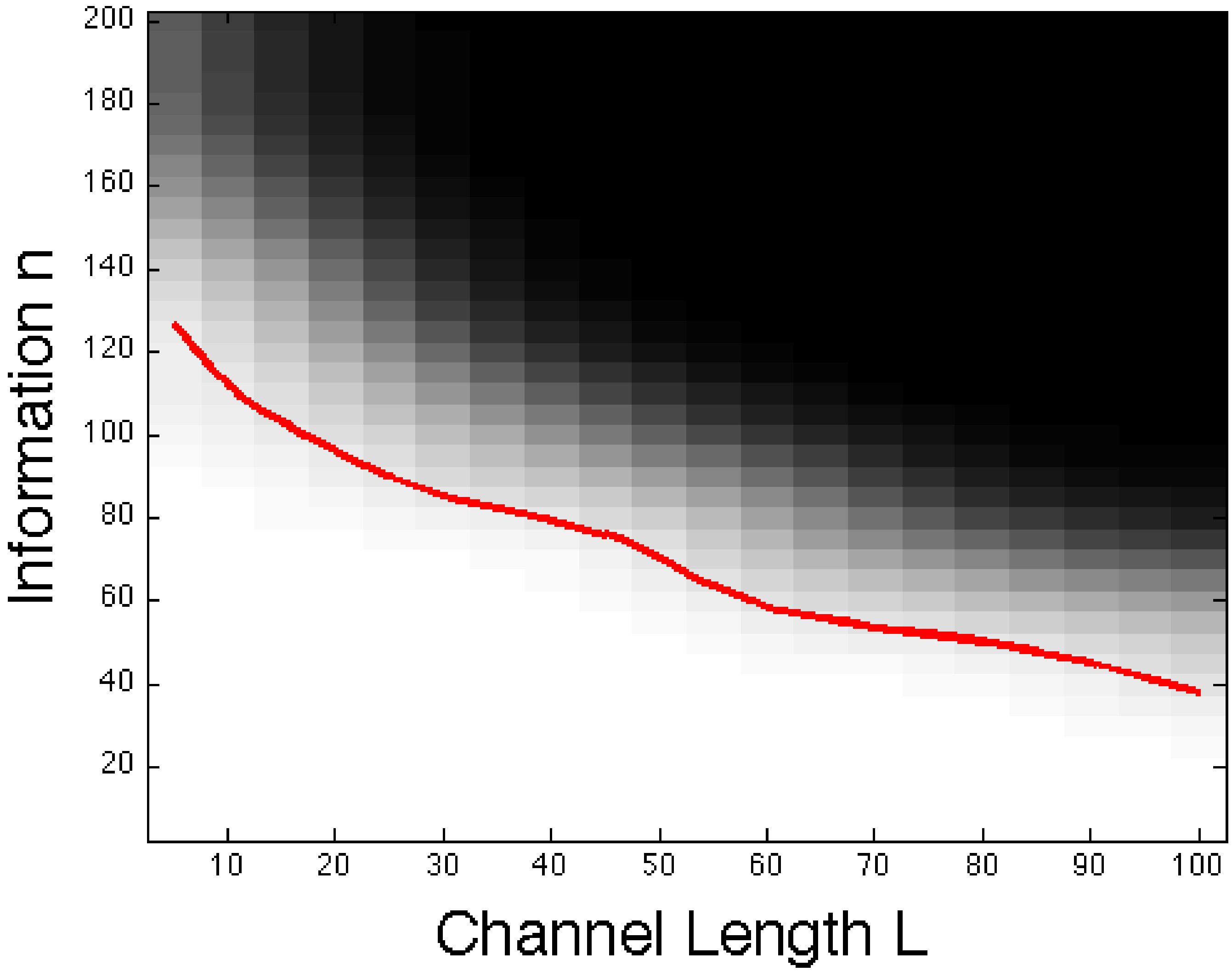
Results: Alternating minimization (m=256, circular convolution model)



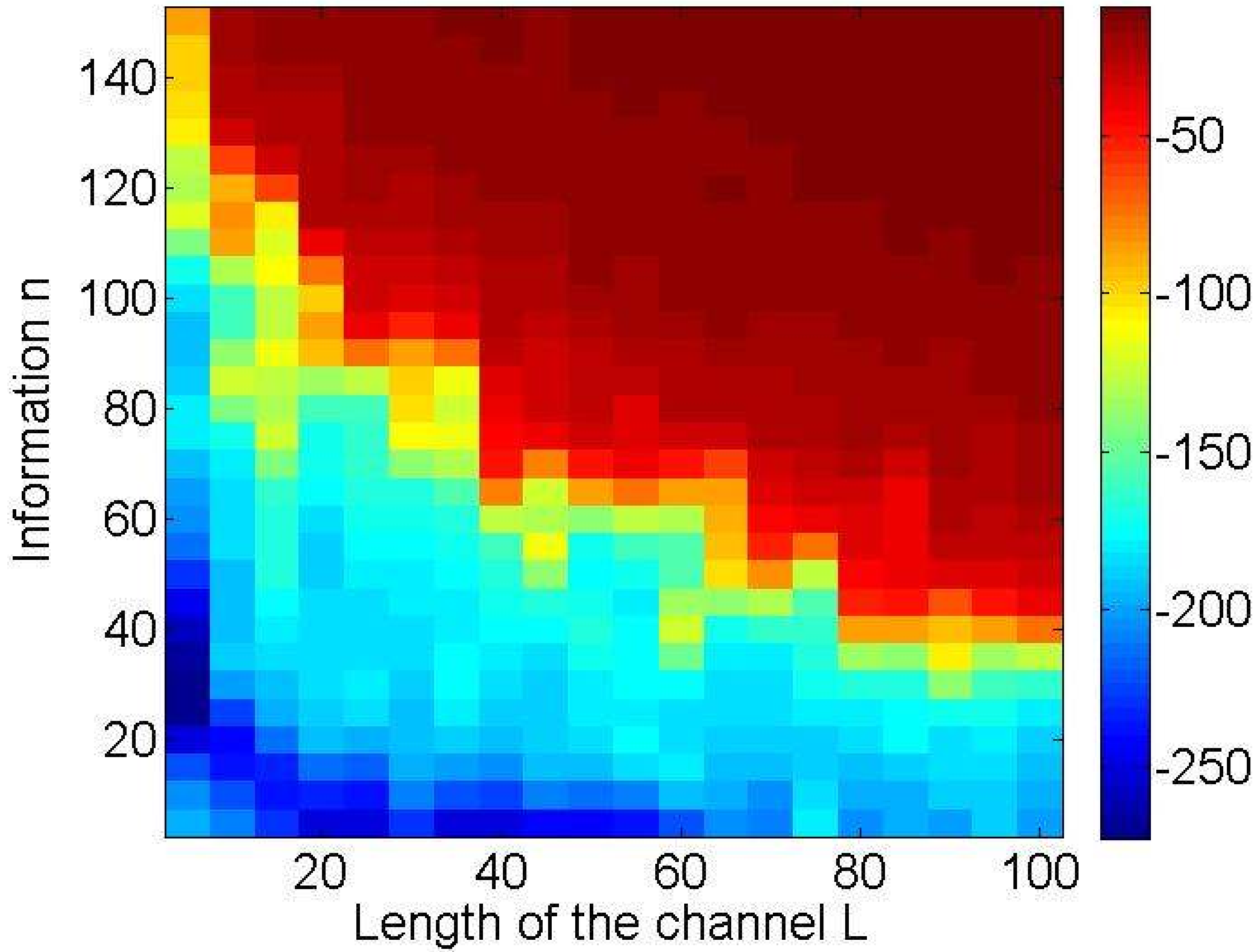
Results: SDP
(m=300)



Results: SDP
(m=300, sparsity = 25%)



Results: SDP (m=256)



Summary

- Random encoding ensures blind deconvolution
- Our proposed algorithms recover the encoded signal from the received signal, without knowledge of the channel.
- Required redundancy scales with length of message and sparsity (and length) of channel
- Nonconvex problems require some more work for global guarantees
- Relax the measurement constraints in the presence of noise

Thankyou !

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Papers:

- *Channel protection: Random coding meets sparse channels, ITW 2009*
- *Random coding and blind deconvolution, Allerton 2009*

Questions