ECE 4601 Homework # 1 Solutions

1. $$r(t) = \alpha \cos 2\pi f_o t + \beta \cos(2\pi(f_o + f_d)t)$$

$$= \alpha \cos 2\pi f_o t + \beta \cos(2\pi f_o t + 2\pi f_d t)$$

$$= \alpha \cos 2\pi f_o t + \beta \cos 2\pi f_o t \cos 2\pi f_d t - \beta \sin 2\pi f_o t \sin 2\pi f_d t$$

$$= (\alpha + \beta \cos 2\pi f_d t) \cos 2\pi f_o t - (\beta \sin 2\pi f_d t) \sin 2\pi f_o t \quad (1)$$

However, $r(t)$ has the complex envelope representation

$$r(t) = R_e\{\tilde{r}(t) e^{j2\pi f_o t}\} \qquad \tilde{r}(t) = \tilde{r}_I(t) + j\tilde{r}_Q(t)$$

$$= R_e\{(\tilde{r}_I(t) + j r_Q(t))(\cos 2\pi f_o t + j \sin 2\pi f_o t)\}$$

$$= \tilde{r}_I(t) \cos 2\pi f_o t - \tilde{r}_Q(t) \sin 2\pi f_o t$$

From (1),
$$\tilde{r}_I(t) = \alpha + \beta \cos 2\pi f_d t \quad (2)$$
$$\tilde{r}_Q(t) = \beta \sin 2\pi f_d t$$

Also
$$\tilde{r}(t) = A(t) e^{j\phi(t)}$$

From (2)

$$A(t) = \sqrt{\tilde{r}_I^2(t) + \tilde{r}_Q(t)}$$

$$= \sqrt{(\alpha + \beta \cos 2\pi f_d t)^2 + \beta^2 \sin^2 2\pi f_d t}$$

$$= \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos 2\pi f_d t}$$

$$\phi(t) = \mathrm{Tan}^{-1} \frac{\tilde{r}_Q(t)}{\tilde{r}_I(t)}$$

$$= \mathrm{Tan}^{-1} \frac{\beta \sin 2\pi f_d t}{\alpha + \beta \cos 2\pi f_d t}$$

2. The channel capacity is

$$C = W \log_2 \left(1 + \frac{P}{N_0 W} \right)$$

where $W = 10^4$ Hz

$P = 10^{-12}$ W

$\frac{N_0}{2} = 10^{-19}$ watts/Hz

$$C = 10^4 \log_2 \left(1 + \frac{10^{-12}}{2 \times 10^{-19} \times 10^4} \right)$$

$$= 10^4 \log_2 (501) \qquad \left\{ \log_2(x) = \frac{\log_e(x)}{\log_e(2)} \right.$$

$$= 89.686 \text{ Kb/s} \Rightarrow \text{Yes, bid!}$$

3 a)

$$P(Y=0) = P(Y=0|X=1)P(X=1) + P(Y=0|X=0)P(X=0)$$

$$= (0.1)(0.6) + (0.7)(0.4)$$

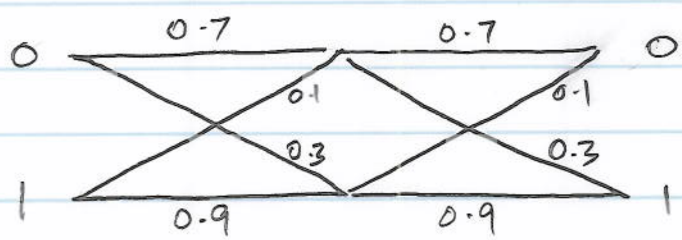
$$= 0.34$$

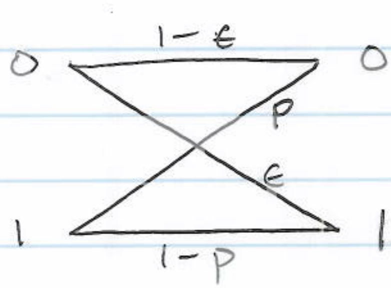
$$P(Y=1) = 1 - P(Y=0) = 0.66$$

b) $$P(X=1|Y=1) = \frac{P(Y=1|X=1)P(X=1)}{P(Y=1|X=0)P(X=0) + P(Y=1|X=1)P(X=1)}$$

$$= \frac{(0.9)(0.6)}{(0.3)(0.4) + (0.9)(0.6)}$$

$$= 0.81818$$

c) 

$\equiv$ 

We have

$$\epsilon = (0.7)(0.3) + (0.3)(0.9) = 0.48$$

$$p = (0.1)(0.7) + (0.9)(0.1) = 0.16$$

We follow the same procedure used for parts a) and b) with

$$P(Y=0 \mid X=0) = 1-\epsilon = 0.52 \qquad P(X=0) = 0.4$$
$$P(Y=0 \mid X=1) = p = 0.16 \qquad P(X=1) = 0.6$$
$$P(Y=1 \mid X=0) = \epsilon = 0.48$$
$$P(Y=1 \mid X=1) = 1-p = 0.84$$

Hence, $P(Y=0) = P(Y=0 \mid X=1)P(X=1) + P(Y=0 \mid X=0)P(X=0)$

$$= (0.16)(0.6) + (0.52)(0.4)$$

$$= 0.304$$

$$P(Y=1) = 1 - P(Y=0) = 0.696$$

$$P(X=1 \mid Y=1) = \frac{(0.84)(0.6)}{(0.48)(0.4) + (0.84)(0.6)}$$

$$= 0.72414$$

4 i) $X = \cos\theta$, $Y = \sin\theta$

$$\mathrm{cov}[X,Y] = E[XY] - E[X]E[Y]$$

$$= E[\cos\theta \sin\theta] - E[\cos\theta]E[\sin\theta]$$

$$= \frac{1}{2} E[\underbrace{\cancel{\sin 2\theta}}_{0}] - E[\underbrace{\cancel{\cos\theta}}_{0}]E[\underbrace{\cancel{\sin\theta}}_{2}]$$

$$= 0$$

e.g. $E[\cos\theta] = \frac{1}{\pi}\int_0^{\pi} \cos\theta \, d\theta = \frac{1}{\pi}\sin\theta \Big|_0^{\pi} = 0$

Hence, X and Y are uncorrelated

ii) X and Y are independent iff

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

or $f_{X|Y}(x|y) = f_X(x)$

But $f_{X|Y}(x|y) = \delta(x - \cos \sin^{-1} y)$
$= \delta(x - \sqrt{1-y^2})$

Also, $X = \cos\theta$, $\theta = \cos^{-1} x$

$f_X(x) = f_\theta(\theta) \left| \frac{d\theta}{dx} \right|$ $\frac{dx}{d\theta} = -\sin\theta \rightarrow \frac{d\theta}{dx} = \frac{-1}{\sin}$

$$= \frac{1}{\pi} \left| \frac{1}{\sin\theta} \right| = \frac{1}{\pi} \left| \frac{1}{\sin \cos^{-1} x} \right|$$

$$= \begin{cases} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} , & -1 \le x \le 1 \\ 0 , & \text{else} \end{cases}$$

Clearly $f_{X|Y}(y|x) \neq f_X(x)$ $\therefore$ not independent.

5. $\mu_y = E[y] = E[a x_1 + b x_2]$

$$= a E[x_1] + b E[x_2]$$

$$= a \mu_{x_1} + b \mu_{y_1}$$

$$\sigma_y^2 = E[(y - \mu_y)^2]$$

$$= E[y^2] - \mu_y^2$$

$$E[y^2] = E[(a x_1 + b x_2)^2]$$

$$= E[a^2 x_1^2 + 2ab x_1 x_2 + b^2 x_2^2]$$

$$= a^2 E[x_1^2] + 2ab E[x_1 x_2] + b^2 E[x_2^2]$$

$$= a^2 E[x_1^2] + 2ab E[x_1] E[x_2] + b^2 E[x_2^2]$$

$$= a^2 E[x_1^2] + 2ab \mu_{x_1} \mu_{x_2} + b^2 E[x_2^2]$$

Then

$$\sigma_y^2 = a^2 E[x_1^2] + 2ab \mu_{x_1} \mu_{x_2} + b^2 E[x_2^2] - a^2 \mu_{x_1}^2 - 2ab \mu_{x_1} \mu_{x_2} - b^2 \mu_{x_2}^2$$

$$= a^2 [E[x_1^2] - \mu_{x_1}^2] + b^2 [E[x_2^2] - \mu_{x_2}^2]$$

$$= a^2 \sigma_{x_1}^2 + b^2 \sigma_{x_2}^2$$

6. a) For $y = x^2$, we have (any probability text)

$$f_y(y) = \frac{f_x(\sqrt{y}) + f_x(-\sqrt{y})}{2\sqrt{y}}$$

Using

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$$

$$f_y(y) = \left[\frac{1}{\sqrt{2\pi}\,\sigma} e^{-y/2\sigma^2} + \frac{1}{\sqrt{2\pi}\,\sigma} e^{-y^2/2\sigma^2}\right]\frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi y}\,\sigma} e^{-y/2\sigma^2}, \quad y \geq 0$$

b) For the limiter, first note

$$\alpha = \text{Prob}(x \geq b) = \text{Prob}(x \leq -b)$$

$$= Q\left(\frac{b}{\sigma}\right)$$

Then

$$f_y(y) = \alpha\,\delta(y+b) + \frac{1}{\sqrt{2\pi}\,\sigma} e^{-y^2/2\sigma^2} + \alpha\,\delta(y-b)$$

$$-b \leq y \leq b$$

c) For hard limiter, note that

$$P(x \geq 0) = P(x \leq 0) = 1/2$$

Hence,

$$f_y(y) = \frac{1}{2}\delta(y+a) + \frac{1}{2}\delta(y-a)$$