ECE 4601 Homework #2 Solutions

1. Note first that $\underline{Y}$ is a column vector

mean

$$\underline{\mu}_y = E[\underline{Y}] = E[A\underline{X}] = A\underline{\mu}_x$$

covariance

$$\Lambda_y = E[(\underline{Y}-\underline{\mu}_y)(\underline{Y}-\underline{\mu}_y)^T]$$

$$= E[A(\underline{X}-\underline{\mu}_x)[A(\underline{X}-\underline{\mu}_x)]^T]$$

$$= E[A(\underline{X}-\underline{\mu}_x)(\underline{X}-\underline{\mu}_x)^T A^T]$$

$$= A\, E[(X-\underline{\mu}_x)(X-\underline{\mu}_x)^T]\, A^T$$

$$= A \Lambda_x A^T$$

2. a)

$$\mu_x(t) = E[X(t)]$$

$$= A\int_0^{\pi/2} \cos(2\pi f_o t + \theta)\,\frac{2}{\pi}\,d\theta$$

$$= \frac{2A}{\pi}\sin(2\pi f_o t + \theta)\Big|_0^{\pi/2}$$

$$= \frac{2A}{\pi}\sin(2\pi f_o t + \pi/2) - \frac{2A}{\pi}\sin(2\pi f_o t)$$

$$= \frac{2A}{\pi}\left[\cos(2\pi f_o t) - \sin 2\pi f_o t\right]$$

$$\sigma_X^2 = E[X^2(t)]$$

$$= \frac{2A^2}{\pi}\int_0^{\pi/2} \cos^2(2\pi f_o t + \theta)\, d\theta$$

$$= \frac{2A^2}{\pi}\int_0^{\pi/2} \left[\frac{1}{2} + \frac{1}{2}\cos(4\pi f_o t + 2\theta)\right] d\theta$$

$$= \frac{A^2}{\pi}\theta \Big|_0^{\pi/2} + \frac{A^2}{2\pi}\sin(4\pi f_o t + 2\theta)\Big|_0^{\pi/2}$$

$$= \frac{A^2}{2} + \frac{A^2}{2\pi}\left[\sin(4\pi f_o t + \pi) - \sin(4\pi f_o t)\right]$$

$$= \frac{A^2}{2} + \frac{A^2}{2\pi}\left[\sin(4\pi f_o t)\cos\pi + \cos(4\pi f_o t)\sin\pi - \sin(4\pi f_o t)\right]$$

$$= \frac{A^2}{2} - \frac{A^2}{\pi}\sin 4\pi f_o t$$

b) $\phi_{XX}(t, t+T) = E[X(t)X(t+\tau)]$

$$= E[A^2\cos(2\pi f_o t + \theta)\cos(2\pi f_o(t+\tau) + \theta)]$$

$$= \frac{A^2}{2}E[\cos 2\pi f_o\tau + \cos(4\pi f_o t + 2\pi f_o\tau + 2\theta)]$$

$$= \frac{A^2}{2}\cos 2\pi f_o\tau + \frac{A^2}{\pi}\int_0^{\pi/2}\cos(4\pi f_o t + 2\pi f_o\tau + 2\theta)\, d\theta$$

$$= \frac{A^2}{2}\cos 2\pi f_o\tau + \frac{A^2}{2\pi}\sin(4\pi f_o t + 2\pi f_o\tau + 2\theta)\Big|_0^{\pi/2}$$

$$= \frac{A^2}{2}\cos 2\pi f_o\tau + \frac{A^2}{2\pi}\sin(4\pi f_o t + 2\pi f_o\tau + \pi) - \frac{A^2}{2\pi}\sin(4\pi f_o t + 2\pi f_o\tau)$$

$$= \frac{A^2}{2}\cos 2\pi f_o\tau - \frac{A^2}{\pi}\sin(4\pi f_o t + 2\pi f_o\tau)$$

c) Not WSS since $\mu_x(t)$ and $\phi_{xx}(t, t+\tau)$ are functions of t.

Cyclostationary, Yes with $T = \frac{1}{f_0}$

$$\mu_x(t+T) = \frac{2A}{\pi}\left(\cos 2\pi f_0\left(t+\frac{1}{f_0}\right) - \sin 2\pi f_0\left(t+\frac{1}{f_0}\right)\right)$$

$$= \frac{2A}{\pi}\left(\cos 2\pi f_0 t - \sin 2\pi f_0 t\right)$$

$$= \mu_x(t)$$

$$\phi_{xx}(t+T, t+\tau+T)$$

$$= \frac{A^2}{2}\cos 2\pi f_c\tau - \frac{A^2}{\pi}\sin\left(4\pi f_c\left(t+\frac{1}{f_0}\right) + 2\pi f_c\tau\right)$$

$$= \frac{A^2}{2}\cos 2\pi f_c\tau - \frac{A^2}{\pi}\sin\left(4\pi f_c t + 2\pi f_c\tau\right)$$

$$= \phi_{xx}(t, t+\tau)$$

1a) $Z(t) = cX(t)Y(t) + d$

$$\phi_{ZZ}(\tau) = E[Z(t)Z(t+\tau)]$$
$$= E[(cX(t)Y(t)+d)(cX(t+\tau)Y(t+\tau)+d)]$$
$$= c^2 E[X(t)Y(t)X(t+\tau)Y(t+\tau)]$$
$$+ cd\,E[X(t)Y(t)]$$
$$+ cd\,E[X(t+\tau)Y(t+\tau)]$$
$$+ d^2$$
$$= c^2 E[X(t)X(t+\tau)]E[Y(t)Y(t+\tau)] + d^2$$
$$= c^2 \phi_{XX}(\tau)\phi_{YY}(\tau) + d^2$$

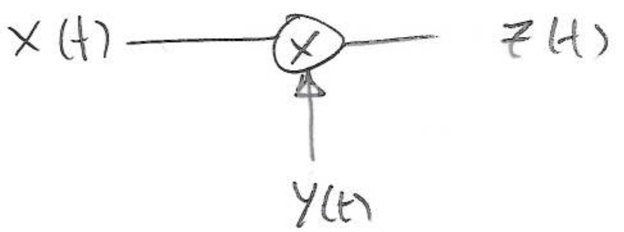
$$\mu_Z = E[cX(t)Y(t)+d] = d$$

Hence, $Z(t)$ is WSS.

b) $Z(t) = X(t)\cos 2\pi f_c t + Y(t)\sin 2\pi f_c t$

$$\phi_{ZZ}(\tau) = E[Z(t)Z(t+\tau)]$$
$$= E[(X(t)\cos 2\pi f_c t + Y(t)\sin 2\pi f_c t)(X(t+\tau)\cos 2\pi f_c(t+\tau) + Y(t+\tau)\sin 2\pi f_c(t+\tau))]$$
$$= E[X(t)X(t+\tau)]\cos 2\pi f_c t\cos 2\pi f_c(t+\tau)$$
$$+ E[Y(t)Y(t+\tau)]\sin 2\pi f_c t\sin 2\pi f_c(t+\tau)$$
$$= \phi_{XX}(\tau)\left[\tfrac{1}{2}\cos 2\pi f_c\tau + \tfrac{1}{2}\cos 2\pi f_c(2t+\tau)\right]$$
$$+ \phi_{YY}(\tau)\left[\tfrac{1}{2}\cos 2\pi f_c\tau - \tfrac{1}{2}\cos 2\pi f_c(2t+\tau)\right]$$
$$= \frac{\phi_{XX}(\tau)+\phi_{YY}(\tau)}{2}\cos 2\pi f_c\tau$$
$$+ \frac{\phi_{XX}(\tau)-\phi_{YY}(\tau)}{2}\cos 2\pi f_c(2t+\tau)$$

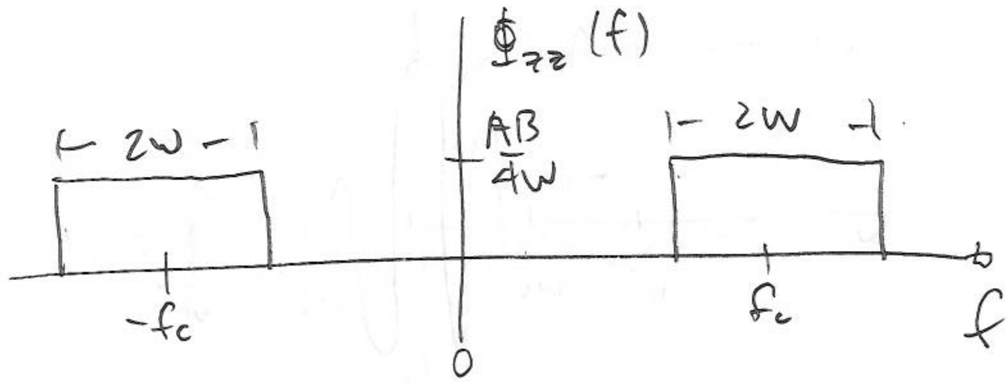
Since $\phi_{ZZ}(\tau)$ depends on t, $Z(t)$ not WSS



$$Z(t) = X(t)\,Y(t)$$

$$\begin{aligned}
\phi_{zz}(\tau) &= E\left[Z(t)\,Z(t+\tau) \right] \\
&= E\left[X(t)\,X(t+\tau)\,Y(t)\,Y(t+\tau) \right] \\
&= E\left[X(t)\,X(t+\tau) \right] E\left[Y(t)\,Y(t+\tau) \right] \\
&= \phi_{xx}(\tau)\,\phi_{yy}(\tau) \\
&= AB \,\text{sinc}\, 2W\tau \cos 2\pi f_0 \tau
\end{aligned}$$

$$\Phi_{zz}(f) = \frac{AB}{4W}\left[\text{rect}\left(\frac{f-f_c}{2W}\right) + \text{rect}\left(\frac{f+f_c}{2W}\right) \right]$$



4/ $N_0 = 1$ watt/Hz

$$w(t) \longrightarrow \boxed{h(t)} \longrightarrow n(t)$$

$$\mu_n = \mu_w H(0) = 0 \quad \text{since } \mu_w = 0$$

$$\mu_{nn}(\tau) = \phi_{nn}(\tau) - \mu_n^2 = \phi_{nn}(\tau)$$

$$S_{nn}(f) = \frac{N_0}{2} |H(f)|^2 = 0.5 \, |H(f)|^2$$

$$H(f) = \frac{1}{10 + j2\pi f}$$

$$S_{nn}(f) = \frac{0.5}{100 + (2\pi f)^2}$$

$$S_{nn}(f) = \frac{0.5}{20} \frac{20}{100 + (2\pi f)^2}$$

$$\phi_{nn}(\tau) = \frac{1}{40} e^{-10|\tau|} = \mu_{nn}(\tau)$$

At time t_0 $n_1 = n(t_0) \sim N(0, 1/40)$

since $\sigma_n^2 = \phi_{nn}(0) = 1/40$

Joint pdf is Gaussian

$n_1 = n(t_0)$

$n_2 = n(t_0 + 0.1)$

$$\underline{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Lambda = \frac{1}{40} \begin{bmatrix} 1 & e^{-1} \\ e^{-1} & 1 \end{bmatrix}$$