Spring 2010
ECE 4601: Assignment 2

- Date Assigned: January 26, 2010.
- Date Due: February 4, 2010.

1. Let the random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$ be jointly Gaussian with mean vector $\boldsymbol{\mu}_X = (\mu_1, \mu_2, \ldots, \mu_n)^T$ and covariance matrix $\mathbf{\Lambda}_X$.

   Define a new random vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)^T$, according to the matrix multiplication
   $$\mathbf{Y} = \mathbf{AX}$$
   where $\mathbf{A}$ is a constant matrix. The vector $\mathbf{Y}$ will be jointly Gaussian as well. Find the mean vector $\boldsymbol{\mu}_Y$ and covariance matrix $\mathbf{\Lambda}_Y$ of the random vector $\mathbf{Y}$.

   Hint: Use the definition of the covariance matrix $\mathbf{\Lambda}$*, the matrix transpose property* $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$*, and the expectation operator* $\mathrm{E}[\ ]$*. No integrals are required!*

2. A random process is defined as
   $$X(t) = A\cos(2\pi f_o t + \Theta)\ , \quad \infty \le t \le \infty$$
   where A and f_o are constants.

   The random phase Θ has the probability density function
   $$f_\Theta(\theta) = \begin{cases} \frac{2}{\pi}, & 0 \le \theta \le \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find the mean and variance of this random process.
   (b) Obtain its autocorrelation function
   (c) Is the process wide-sense stationary? Cyclostationary (period wide-sense stationary)?

3. Suppose $X(t)$ and $Y(t)$ are zero-mean, wide sense stationary, continuous-time random processes. If $X(t)$ and $Y(t)$ are independent, find the autocorrelation function for $Z(t)$ in terms of the autocorrelation functions for $X(t)$ and $Y(t)$ in each of the cases that follow. In each case, determine whether or not the random process $Z(t)$ is wide-sense stationary.

(a) $Z(t) = cX(t)Y(t) + d$, where c and d are deterministic constants.

(b) $Z(t) = X(t)\cos(2\pi f_c t) + Y(t)\sin(2\pi f_c t)$.

4. Obtain the autocorrelation functions and power spectral densities of the outputs of the following systems $h(t) \leftrightarrow H(f)$ having the input autocorrelation functions $\phi_{XX}(\tau)$ or power spectral densities $\Phi_{XX}(f)$ as the case may be.

   (a)

   $$H(f) = \mathrm{rect}(f/2B) \qquad \phi_{XX}(\tau) = \frac{N_o}{2}\delta(\tau)$$

   where B and N_o are positive constants.

   (b)

   $$h(t) = Ae^{-\alpha t}u(t) \qquad \Phi_{XX}(f) = \frac{B}{1 + (2\pi\beta f)^2}$$

   where A, B, β, and α are positive constants.

5. The two statistically independent inputs to a multiplier have autocorrelation function

   $$\begin{aligned} \phi_{XX}(\tau) &= A\cos 2\pi f_o \tau \\ \phi_{YY}(\tau) &= B\mathrm{sinc}(2W\tau) \end{aligned}$$

   Obtain and plot the power spectrum of the multiplier output.

6. The input to a low pass filter with impulse response

   $$h(t) = \exp(-10t)u(t)$$

   is white, Gaussian noise with single-sided power spectral density 1 watts/Hz. Obtain the following:

   (a) The mean of the output

   (b) The autocovariance function of the output

   (c) The pdf of the output at a single time instant t_o

   (d) The joint pdf of the output at time instants t_o and $t_o + 0.1$s