EE4601
Communication Systems

Lecture 11
Nyquist Pulse Shaping

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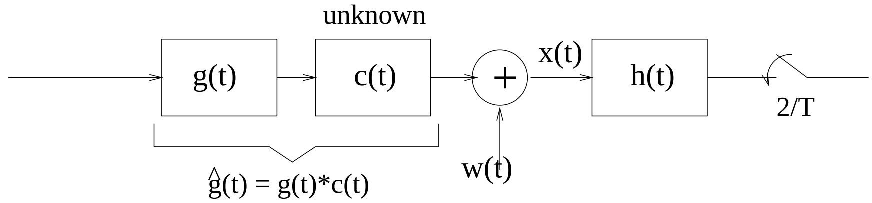
Matched Filtering and Pulse Shaping



- To maximize the signal-to-noise ratio at the output of the receiver filter $h(t)$, in theory we match the receiver filter to the received pulse $\hat{g}(t) = g(t) * c(t)$, i.e., $h(t) = \tilde{g}(T - t)$. However, if $c(t)$ is unknown, then so is $h(t)$.
- Practical Solution: Choose $h(t)$ matched to the transmitted pulse $g(t)$, i.e., choose $h(t) = g(T - t)$, over-sample by a factor of 2, and process 2 samples per baud interval.
  - This is optimal, similar to the case when $c(t)$ is known, but the proof is beyond the scope of this course.

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Matched Filtering and Pulse Shaping

- To design the transmit and receiver filters, we will assume an ideal channel $c(t) = \delta(t)$, so that the overall pulse (ignoring time delay) is

$$\begin{aligned} p(t) &= g(t) * h(t) \\ &= g(t) * g(-t) \end{aligned}$$

- Taking the Fourier transform of both sides

$$P(f) = G(f)G^*(f) = |G(f)|^2$$

- Hence

$$|G(f)| = \sqrt{|P(f)|}$$

- For many practical pulses, $g(t)$, we will also see that $g(t) = g(-t)$, i.e., the pulse is even in t, so that $h(t) = g(t)$.

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Conditions for ISI free transmission

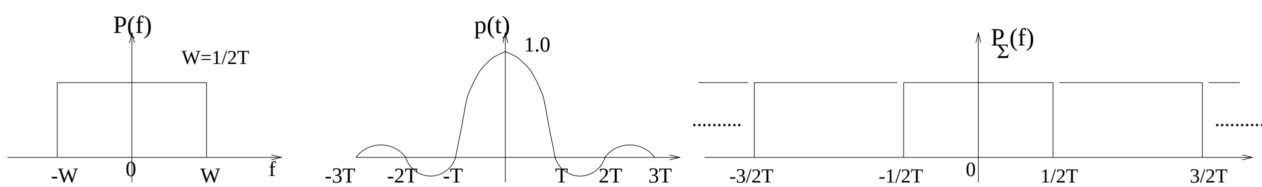
The condition for ISI-free transmission is

$$p_k = \delta_{k0} p_0 = \begin{cases} p_0 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

That is, $p(t)$ must have equally spaced zero crossings, separated by T seconds.
Theorem: The pulse $p(t)$ satisfies $p_k = \delta_{k0} p_0$ iff

$$P_{\sum}(f) \triangleq \frac{1}{T} \sum_{n=-\infty}^{\infty} P(f + n/T) = p_0$$

That is the folded spectrum $P_{\sum}(f)$ is flat.



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ISI free transmission

<u>Proof:</u>

$$\begin{aligned} p_k &= \int_{-\infty}^{\infty} P(f) e^{j2\pi fkT} df \\ &= \sum_{n=-\infty}^{\infty} \int_{(2n-1)/2T}^{(2n+1)/2T} P(f) e^{j2\pi fkT} df \quad f' = f - n/T \\ &= \sum_{n=-\infty}^{\infty} \int_{-1/2T}^{1/2T} P(f' + n/T) e^{j2\pi k(f'+n/T)T} df' \\ &= \int_{-1/2T}^{1/2T} e^{j2\pi f' kT} \left[\sum_{n=-\infty}^{n=\infty} P(f' + n/T) \right] df' \end{aligned} \tag{1}$$

To prove sufficiency, we assume that $\sum_{n=-\infty}^{\infty} P(f' + n/T) = p_0 T$ is true. Then,

$$p_k = p_0 T \int_{-1/2T}^{1/2T} e^{j2\pi f' kT} df' = \frac{\sin \pi k}{\pi k} p_0 = \delta_{k0} p_{k0}$$

To prove necessity, we have from (1)

$$p_k = T \int_{-1/2T}^{1/2T} P_\Sigma(f') e^{j2\pi f' kT} df'$$

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Nyquist Pulse

Hence, p_k and $P_\Sigma(f)$ are a Fourier series pair, i.e.,

$$P_\Sigma(f) = \sum_{k=-\infty}^{\infty} p_k e^{j2\pi fkT}$$

If $p_k = p_0\delta_{k0}$ is assumed true, then from the above equation $P_\Sigma(f) = p_0$.

- Nyquist Pulse Shaping: A pulse $p(t)$ that yields zero-ISI is one having a folded spectrum that is flat.
  - The pulse $p(t)$ can be generated by choosing $P(f)$ as shown on the following slide.

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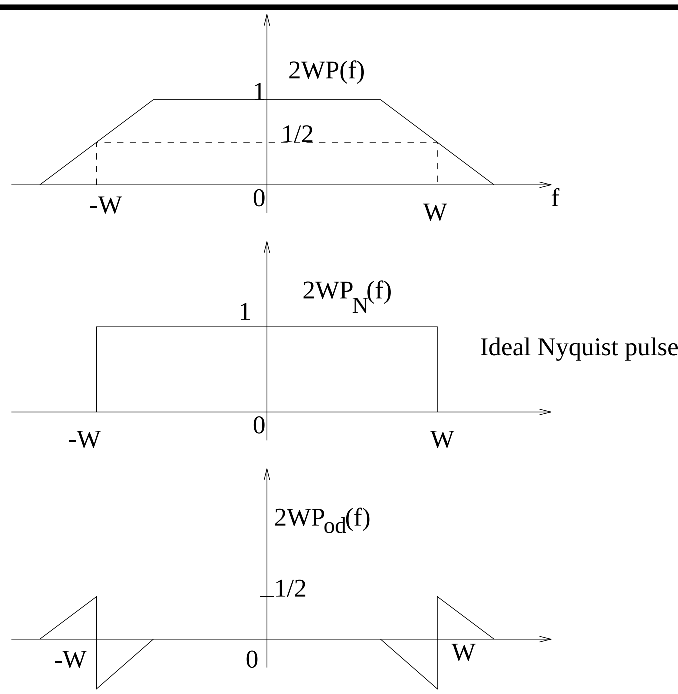
Nyquist Pulse Shaping



Note $P(f) = P_N(f) + P_{\text{od}}(f)$.
$P_{\text{od}}(f)$ can be <u>any</u> function that has skew symmetry about $f = W$.

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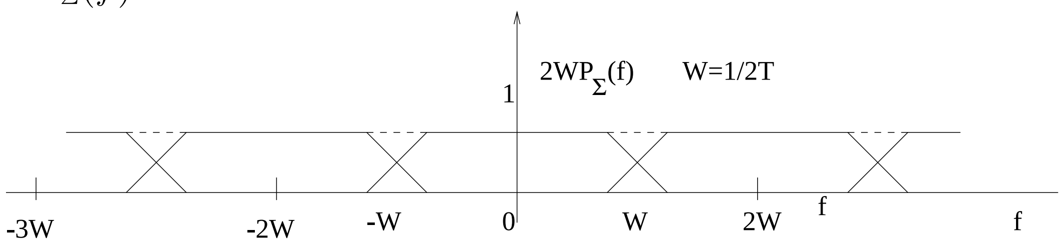
Nyquist Pulse

Note that $P_\Sigma(f)$ is flat under this condition.



<u>Example: Raised Cosine</u>

$$2WP_{\rm od}(f) = \begin{cases} -\frac{1}{2} - \frac{1}{2}\sin\frac{\pi(|f|-W)}{2f_x} & W - f_x \le |f| \le W \\ \frac{1}{2} - \frac{1}{2}\sin\frac{\pi(|f|-W)}{2f_x} & W \le |f| \le W + f_x \end{cases}$$

f_x = bandwidth expansion, $\frac{f_x}{W} \times 100$ = excess bandwidth (%), $\alpha = \frac{f_x}{W}$ = roll off factor

$$2WP(f) = \begin{cases} 1 & 0 \le |f| \le W - f_x \\ \frac{1}{2}\left[1 - \sin\frac{\pi(|f|-W)}{2f_x}\right] & W - f_x \le |f| \le W + f_x \\ 0 & |f| \ge W + f_x \end{cases}$$

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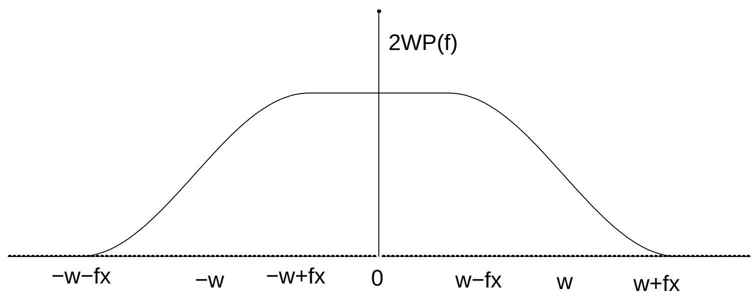
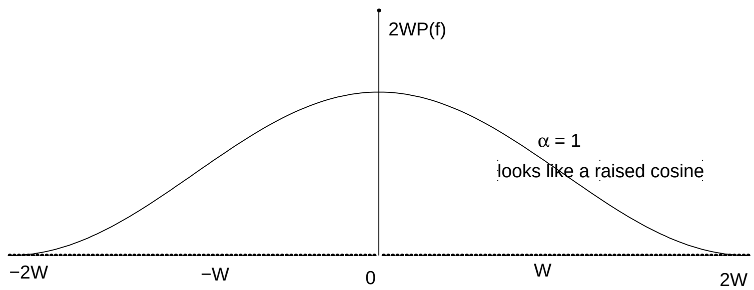
Raised Cosine Pulse





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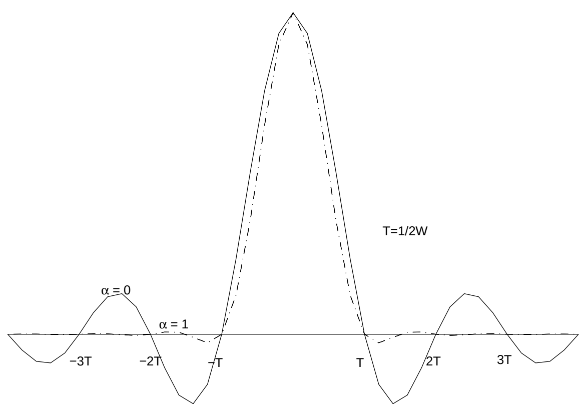
Raised Cosine Impulse Response

Impulse response - Since $P(f)$ is even, the inverse cosine transform yields

$$
\begin{aligned}
p(t) &= 2\int_0^{W+f_x} P(f)\cos 2\pi f t df \\
&= 2.\frac{1}{2W}\int_0^{W-f_x}\cos 2\pi f t df + 2.\frac{1}{2W}\int_{W-f_x}^{W+f_x}\frac{1}{2}\left[1-\sin\frac{\pi|f|-W}{2f_x}\right]\cos 2\pi f t df \\
&= \frac{\sin 2\pi W t}{2\pi W t}.\frac{\cos 2\pi f_x t}{1-(4f_x t)^2}
\end{aligned}
$$



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Square Root Raised Cosine Pulse

- To implement a matched filter, we split the overall pulse $P(f)$ between the transmit and receive filters, i.e., $p(t) = g(t) * g(-t)$.
- We have seen earlier that $P(f) = G(f)G^*(f) = |G(f)|^2$, so that $|G(f)| = \sqrt{P(f)}$.
- With square-root raised cosine pulse shaping

$$\sqrt{2W}|G(f)| = \begin{cases} 1 & 0 \leq |f| \leq wW - f_x \\ \sqrt{\frac{1}{2}\left[1 - \sin\frac{\pi(|f|-W)}{2f_x}\right]} & W - f_x \leq |f| \leq W + f_x \\ 0 & |f| \geq W + f_x \end{cases}$$

- The impulse response is

$$g(t) = 4\alpha \frac{\cos\left[(1+\alpha)\pi t/T\right] + \sin\left[(1-\alpha)\pi t/T\right](4\alpha t/T)^{-1}}{\pi\sqrt{T}\left[1 - 16\alpha^2 t^2/T^2\right]}$$

where $\alpha = f_x/W$.

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