

EE4601

Communication Systems

Lecture 16

Orthogonal Expansions

Basic Problem

Problem:

Suppose that we have a set of M finite energy signals $S = \{s_1(t), s_2(t), \dots, s_M(t)\}$, where each signal has a duration T seconds.

Every T seconds one of the waveforms from the set S is selected for transmission over an AWGN channel. The transmitted waveform is

$$x(t) = \sum_n s_n(t - nT)$$

The received noise corrupted waveform is

$$r(t) = \sum_n s_n(t - nT) + n(t)$$

By observing $r(t)$ we wish to determine the *time sequence* of waveforms $\{s_n(t)\}$ that was transmitted. That is, in each T second interval, we must determine which $s_i(t) \in S$ was transmitted.

Orthogonal Expansions

Consider a real signal $s(t)$ with finite energy E_s ,

$$E_s = \int_{-\infty}^{\infty} s^2(t) dt$$

Suppose there exists a set of *orthonormal* functions $\{f_n(t)\}, n = 1, \dots, N$. By orthonormal we mean

$$\int_{-\infty}^{\infty} f_n(t) f_k(t) dt = \delta_{kn} \quad \delta_{kn} = \begin{cases} 1 & , \quad k = n \\ 0 & , \quad k \neq n \end{cases}$$

We now approximate $s(t)$ as the weighted linear sum

$$\hat{s}(t) = \sum_{k=1}^N s_k f_k(t)$$

and wish to determine the $s_k, k = 1, \dots, N$ to minimize the square error

$$\begin{aligned} \varepsilon &= \int_{-\infty}^{\infty} (s(t) - \hat{s}(t))^2 dt \\ &= \int_{-\infty}^{\infty} \left(s(t) - \sum_{k=1}^N s_k f_k(t) \right)^2 dt \end{aligned}$$

Orthogonal Expansions

To minimize the mean square error, we take the partial derivative with respect to each of the s_k and set equal to zero, i.e., for the n th term we solve

$$\frac{\partial \varepsilon}{\partial s_n} = 2 \int_{-\infty}^{\infty} \left(s(t) - \sum_{k=1}^N s_k f_k(t) \right) f_n(t) dt = 0.$$

Using the orthonormal property of the basis functions, $s_n = \int_{-\infty}^{\infty} s(t) f_n(t) dt$ and

$$\begin{aligned} \varepsilon &= \int_{-\infty}^{\infty} \left(s(t) - \sum_{k=1}^N s_k f_k(t) \right)^2 dt \\ &= \int_{-\infty}^{\infty} s^2(t) dt - 2 \int_{-\infty}^{\infty} s(t) \sum_{k=1}^N s_k f_k(t) dt + \int_{-\infty}^{\infty} \sum_{k=1}^N s_k f_k(t) \sum_{\ell=1}^N s_\ell f_\ell(t) dt \\ &= \int_{-\infty}^{\infty} s^2(t) dt - 2 \sum_{k=1}^N s_k \int_{-\infty}^{\infty} s(t) f_k(t) dt + \sum_{k=1}^N \sum_{\ell=1}^N s_k s_\ell \int_{-\infty}^{\infty} f_k(t) f_\ell(t) dt \\ &= E_s - \sum_{k=1}^N s_k^2 \end{aligned}$$

For a complete set of basis functions $\varepsilon = 0$.

Gram-Schmidt Orthonormalization

Suppose that we have a set of finite energy real signals $\{s_i(t)\}, i = 1, \dots, M$. We wish to obtain a *complete* set of orthonormal basis functions for the signal set. This can be done in 2 steps.

Step1: Determine if the set of waveforms is linearly independent. If they are linearly dependent, then there exists a set of coefficients a_1, a_2, \dots, a_M , not all zero, such that

$$a_1 s_1(t) + a_2 s_2(t) + \dots + a_M s_M(t) = 0.$$

Suppose, without loss of generality, that $a_M \neq 0$. If $a_M = 0$, then the signal set can be permuted so that $a_M \neq 0$. Then

$$s_M(t) = - \left(\frac{a_1}{a_M} s_1(t) + \frac{a_2}{a_M} s_2(t) + \dots + \frac{a_{M-1}}{a_M} s_{M-1}(t) \right) .$$

Next consider the reduced signal set $\{s_i(t)\}_{i=1}^{M-1}$. If this set of waveforms is linearly dependent, then there exists another set of co-efficients $\{b_i\}_{i=1}^{M-1}$, not all zero, such that

$$b_1 s_1(t) + b_2 s_2(t) + \dots + b_{M-1} s_{M-1}(t) = 0 .$$

Gram-Schmidt Orthonormalization

We continue until a set $\{s_i(t)\}_{i=1}^N$ of linearly independent waveforms is obtained. Note that $N \leq M$ with equality if and only if the set of waveforms $\{s_i(t)\}_{i=1}^M$ is linearly independent.

If $N < M$, then the set of linearly independent waveforms $\{s_i(t)\}_{i=1}^N$ is not unique, but any one will do.

Step 2: From the set $\{s_i(t)\}_{i=1}^N$ construct the set of N orthonormal basis functions $\{f_i(t)\}_{i=1}^N$ as follows. First, let

$$f_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

where E_1 is the energy in the waveform $s_1(t)$, given by

$$E_1 = \int_0^T s_1^2(t) dt$$

Then

$$s_1(t) = \sqrt{E_1} f_1(t) = s_{11} f_1(t)$$

where $s_{11} = \sqrt{E_1}$.

Gram-Schmidt Orthonormalization

Next, by using the waveform $s_2(t)$ we obtain

$$s_{21} = \int_0^T s_2(t) f_1(t) dt$$

along with the intermediate function

$$g_2(t) = s_2(t) - s_{21} f_1(t)$$

Note that $g_2(t)$ is orthogonal to $f_1(t)$.

The second basis function is

$$\begin{aligned} f_2(t) &= \frac{g_2(t)}{\sqrt{\int_0^T (g_2(t))^2 dt}} \\ &= \frac{s_2(t) - s_{21} f_1(t)}{\sqrt{E_2 - s_{21}^2}} \end{aligned}$$

Gram-Schmidt Orthonormalization

Continuing in the above fashion, we define the i th intermediate function

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij} f_j(t)$$

where

$$s_{ij} = \int_0^T s_i(t) f_j(t) dt$$

The set of functions

$$f_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T (g_i(t))^2}} \quad i = 1, 2, \dots, N$$

is the required set of complete orthonormal basis functions.

Gram-Schmidt Orthonormalization

We can now write the signals as weighted linear combinations of the basis functions, i.e.,

$$\begin{aligned}s_1(t) &= s_{11}f_1(t) \\s_2(t) &= s_{21}f_1(t) + s_{22}f_2(t) \\s_3(t) &= s_{31}f_1(t) + s_{32}f_2(t) + s_{33}f_3(t) \\&\vdots = \vdots \\s_N(t) &= s_{N1}f_1(t) + \cdots + s_{NN}f_N(t)\end{aligned}$$

For the remaining signals $s_i(t), i = N + 1, \dots, M$, we have

$$s_i(t) = \sum_{k=1}^N s_{ik}f_k(t)$$

where

$$s_{ik} = \int_0^T s_i(t)f_k(t)dt$$

Signal Vectors

It follows that the signal set $s_i(t), i = 1, \dots, M$ can be expressed in terms of a set of signal vectors $\mathbf{s}_i, i = 1, \dots, M$ in an N -dimensional signal space, i.e.,

$$\begin{aligned} s_1(t) &\leftrightarrow \mathbf{s}_1 = (s_{11}, s_{12}, \dots, s_{1N}) \\ s_2(t) &\leftrightarrow \mathbf{s}_2 = (s_{21}, s_{22}, \dots, s_{2N}) \\ &\quad \vdots = \quad \vdots \\ s_M(t) &\leftrightarrow \mathbf{s}_M = (s_{M1}, s_{M2}, \dots, s_{MN}) \end{aligned}$$