EE4601
Communication Systems

Lecture 24

Noncoherent Detection Error Probability

Rayleigh Random Variable

Let:

$$R = \sqrt{X^2 + Y^2},\ \Phi = \mathrm{Tan}^{-1}\frac{Y}{X}$$

$$\text{where } X, Y \sim N(0, \sigma^2)$$

Then:

$$X = R\cos\Phi$$

$$Y = R\sin\Phi$$

$$f_{R,\Phi}(r,\phi) = f_{XY}(r\cos\phi, r\sin\phi)|J|$$

where

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{x^2+y^2}{2\sigma^2}\right\}$$

and

$$|J| = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \phi} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \phi} \end{vmatrix} = \begin{vmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{vmatrix} = r$$

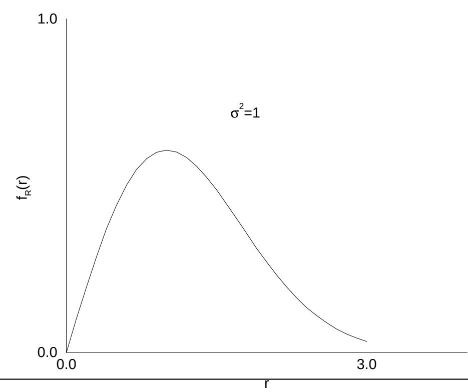
Rayleigh Random Variable

$$f_{R,\Phi}(r,\phi) = \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{r^2}{2\sigma^2}\right\}$$

$$f_R(r) = \int_0^{2\pi} f_{R,\Phi}(r,\phi)d\phi = \frac{r}{\sigma^2}\exp\left\{-\frac{r^2}{2\sigma^2}\right\} \ \ , r \geq 0$$

$f_R(r)$ has a Rayleigh distribution and, for our problem,

$$f_{l_2}(x) = \frac{2x}{N_0}\exp\left\{-\frac{x^2}{N_0}\right\} \ \ , x \geq 0$$



Rician Random Variable

Let:

$$R = \sqrt{X^2 + Y^2},\ \Phi = \mathrm{Tan}^{-1}\frac{Y}{X}$$

$$\text{where } X \sim N(\sqrt{E}\cos\Phi_1, \sigma^2),\ Y \sim N(\sqrt{E}\sin\Phi_1, \sigma^2)$$

Using the same procedure as before,

$$\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{(x-\sqrt{E}\cos\phi_1)^2 + (y-\sqrt{E}\sin\phi_1)^2}{2\sigma^2}\right\} \\
f_{R,\Phi}(r,\phi) &= \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{r^2+E}{2\sigma^2}\right\}\exp\left\{-\frac{\sqrt{E}r}{\sigma^2}\cos(\phi-\phi_1)\right\} \\
f_R(r) &= \frac{r}{\sigma^2}\exp\left\{-\frac{r^2+E}{2\sigma^2}\right\} I_0\left(\frac{\sqrt{E}r}{\sigma^2}\right),\ r \geq 0,
\end{aligned}$$

where

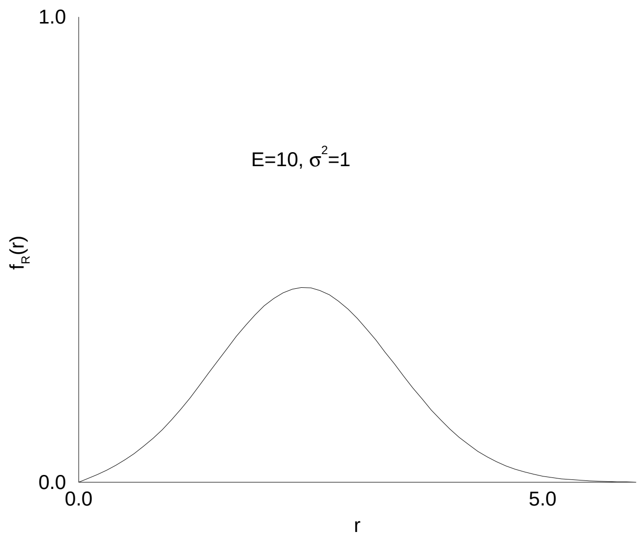
$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi}\exp\{-x\cos\phi\}\,d\phi$$

is a zero-order modified Bessel function of the first kind.

Rician Random Variable

For our problem

$$f_{l_1}(x) = \frac{2x}{N_o} \exp\left\{-\frac{x^2+E}{N_o}\right\} I_0\left(\frac{2\sqrt{E}x}{N_o}\right) \quad , x \geq 0$$



Bit Error Probability

$$\begin{aligned} P_b &= P(l_2 > l_1 | s_1(t) \text{ sent}) \\ &= \int_0^\infty P(l_2 > l_1 | l_1 \text{ and } s_1(t) \text{ sent}) f(l_1) dl_1 \end{aligned}$$

$$\begin{aligned} P(l_2 > l_1 | l_1 \text{ and } s_1(t) \text{ sent}) &= \int_{l_1}^\infty f(l_2) dl_2 \\ &= \exp\left\{-\frac{l_1^2}{N_0}\right\} \end{aligned}$$

$$\begin{aligned} P_b &= \int_0^\infty \exp\left\{-\frac{l_1^2}{N_0}\right\} \frac{2l_1}{N_0} \exp\left\{-\frac{l_1^2 + E}{N_0}\right\} I_0\left(\frac{2l_1\sqrt{E}}{N_0}\right) dl_1 \\ &= \int_0^\infty \frac{2l_1}{N_0} \exp\left\{-\frac{2l_1^2 + E}{N_0}\right\} I_0\left(\frac{2l_1\sqrt{E}}{N_0}\right) dl_1 \end{aligned}$$

Error Probability

Define $v = \frac{2l_1}{\sqrt{N_0}}$. Then,

$$P_b = \frac{1}{2} \exp\left\{-\frac{E}{2N_0}\right\} \int_0^\infty v \exp\left\{-\frac{v^2 + a^2}{2}\right\} I_0\left(av\right) dv$$

where $a = \sqrt{\frac{E}{N_0}}$.

However, the integral is a Rice pdf that is being integrated over its entire range, i.e., the integral is equal to 1.

Therefore,

$$P_b = \tfrac{1}{2} \exp\left\{-\tfrac{E}{2N_0}\right\}$$

ASK Signals

Noncoherent detection can be used for ASK signals as well. Consider the two signals

$$\begin{aligned} s_1(t) &= \sqrt{\frac{2E}{T}} \cos(2\pi f_c t),\ 0 \le t \le T \\ s_2(t) &= 0,\ 0 \le t \le T \end{aligned}$$

We use just a single energy detector at frequency f_c.

If $s_1(t)$ is sent, the pdf of the detector output, ℓ, is

$$f_{\ell|s_1}(x) = \frac{2x}{N_0} \exp\left\{-\frac{x^2}{N_0}\right\} \quad , x \ge 0$$

If $s_2(t)$ is sent, the pdf of the detector output is

$$f_{\ell|s_2}(x) = \frac{2x}{N_o} \exp\left\{-\frac{x^2 + E}{N_o}\right\} I_0\left(\frac{2\sqrt{E}x}{N_o}\right) \quad , x \ge 0$$

ASK Signals

The optimum decision threshold, λ, is where the two conditional pdfs cross. To find the threshold, we solve

$$
\begin{aligned}
f_{\ell|s_1}(x) &= f_{\ell|s_2}(x) \\
\frac{2x}{N_0}\exp\left\{-\frac{x^2}{N_0}\right\} &= \frac{2x}{N_o}\exp\left\{-\frac{x^2+E}{N_o}\right\} I_0\left(\frac{2\sqrt{E}x}{N_o}\right) \\
1 &= \exp\left\{-\frac{E}{N_o}\right\} I_0\left(\frac{2\sqrt{E}x}{N_o}\right)
\end{aligned}
$$

The solution to the above equation gives λ_{opt}.

ASK Signals

Then

$$\begin{aligned} P_b &= \frac{1}{2}\int_{\lambda_{\text{opt}}}^{\infty} f_{\ell|s_1}(x)dx + \frac{1}{2}\int_{0}^{\lambda_{\text{opt}}} f_{\ell|s_2}(x)dx \\ &= \frac{1}{2}\exp\left\{-\frac{{\lambda_{\text{opt}}}^2}{N_0}\right\} + \frac{1}{2}\left[1 - Q\left(\sqrt{\frac{2E}{N_o}}, \sqrt{\frac{2{\lambda_{\text{opt}}}^2}{N_o}}\right)\right] \end{aligned}$$

where $Q(x, y)$ is called a Marcum-Q function, and

$$Q\left(\sqrt{\frac{2E}{N_o}}, \sqrt{\frac{2{\lambda_{\text{opt}}}^2}{N_o}}\right) = \int_{\lambda_{\text{opt}}}^{\infty} \frac{2x}{N_o}\exp\left\{-\frac{x^2+E}{N_o}\right\} I_0\left(\frac{2\sqrt{E}x}{N_o}\right) dx$$

Unfortunately, the result does not exist in closed form.