EE4601
Communication Systems

Lecture 6

Ergodic Random Processes,
Power Spectrum

Ergodic Random Processes

An **ergodic** random process is one where time averages are equal to ensemble averages. Hence, for all $g(\mathbf{X})$ and $\mathbf{X}$

$$\begin{aligned} \mathrm{E}[g(\mathbf{X})] &= \int_{-\infty}^{\infty} g(\mathbf{X}) p_{\mathbf{X}(t)}(\mathbf{x}) d\mathbf{x} \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g[\mathbf{X}(t)] \mathrm{d}t \\ &= < g[\mathbf{X}(t)] > \end{aligned}$$

For a random process to be ergodic, it must be strictly stationary. However, not all strictly stationary random processes are ergodic.

A random process is **ergodic in the mean** if

$$< X(t) >= \mu_X$$

and **ergodic in the autocorrelation** if

$$< X(t)X(t+\tau) >= \phi_{XX}(\tau)$$

Example (cont'd)

Recall the random process

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

where A and f_c are constants, and Θ is a uniformly distributed random phase.

$$p_\Theta(\theta) = \begin{cases} 1/(2\pi)\ , & 0 \le \theta \le 2\pi \\ 0\ , & \text{elsewhere} \end{cases}$$

The time average mean of $X(t)$ is

$$< X(t) >= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A\cos(2\pi f_c t + \theta)dt = 0$$

In this example $\mu_X(t) =< X(t) >$, so the random process $X(t)$ is ergodic in the mean.

N.B. Make sure you understand the difference between the *time average* and *ensemble average*.

Example (cont'd)

The time average autocorrelation of $X(t)$ is

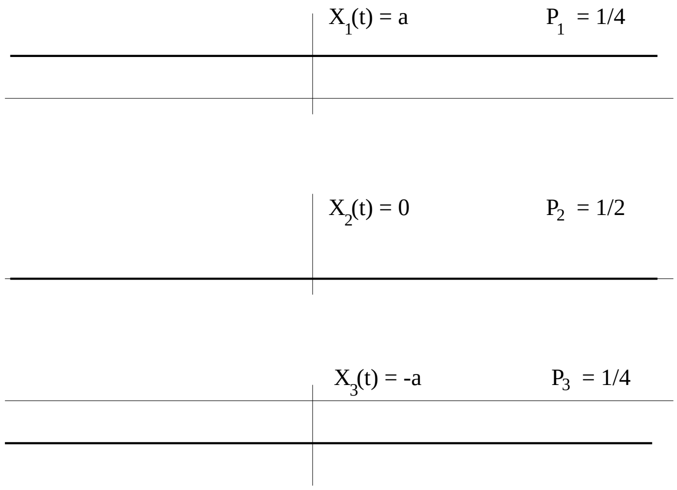
$$
\begin{aligned}
< X(t)X(t+\tau) > &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A^2 \cos(2\pi f_c t + 2\pi f_c \tau + \theta) \cos(2\pi f_c t + \theta) dt \\
&= \lim_{T\to\infty} \frac{A^2}{4T} \int_{-T}^{T} A^2 \left[\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + \theta)\right] dt \\
&= \frac{A^2}{2} \cos(2\pi f_c \tau)
\end{aligned}
$$

The random process $X(t)$ is ergodic in the autocorrelation.

It follows that the random process $X(t)$ in this example is *ergodic in the mean and autocorrelation.*

Example

Consider the random process shown below.



Example (cont'd)

For this example, the *ensemble* and *time average* means are, respectively,

$$\mu_X = \mathrm{E}[X(t)] = 0$$

$$\langle X(t)\rangle = \begin{cases} a & \text{with probability } 1/4 \\ 0 & \text{with probability } 1/2 \\ -a & \text{with probability } 1/4 \end{cases}$$

Hence, $X(t)$ is *not ergodic in the mean.*

The *ensemble* and *time average* autocorrelations are

$$\phi_{XX}(\tau) = \mathrm{E}[X(t)X(t+\tau)] = a^2(1/4) + 0(1/2) + (-a)^2(1/4) = a^2/2$$

$$\langle X(t)X(t+\tau)\rangle = \begin{cases} a^2 & \text{with probability } 1/2 \\ 0 & \text{with probability } 1/2 \end{cases}$$

Hence, $X(t)$ is *not ergodic in the autocorrelation.*

Example (cont'd)

Note that

$$\begin{aligned} \mathrm{E}[\langle X(t) \rangle] &= \mu_X \\ \mathrm{E}[\langle X(t)X(t+\tau) \rangle] &= \phi_{XX}(\tau) \end{aligned}$$

Because of this property $\langle X(t) \rangle$ and $\langle X(t)X(t+\tau) \rangle$ are said to provide *unbiased estimates* of μ_X and $\phi_{XX}(\tau)$, respectively.

Power Spectral Density

The power spectral density (psd) of a random process $X(t)$ is the Fourier transform of its autocorrelation function, i.e.,

$$\begin{aligned}\Phi_{XX}(f) &= = \int_{-\infty}^{\infty} \phi_{XX}(\tau) e^{-j2\pi f\tau} d\tau \\ \phi_{XX}(\tau) &= \int_{-\infty}^{\infty} \Phi_{XX}(f) e^{j2\pi f\tau} df \ .\end{aligned}$$

We have seen that $\phi_{XX}(\tau)$ is real and even. Therefore, $\Phi_{XX}(-f) = \Phi_{XX}(f)$ meaning that $\Phi_{XX}(f)$ is also real and even.

The total power (ac + dc), P, in a random process $X(t)$ is

$$P = \mathrm{E}[X^2(t)] = \phi_{XX}(0) = \int_{-\infty}^{\infty} \Phi_{XX}(f) df$$

a famous result known as **Parseval's theorem**.

Example

$$X(t) = A\cos(2\pi f_c t + \Theta)$$

where A and f_v are constants and

$$p_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}\ , & -\pi \le \theta \le \pi \\ 0\ , & \text{elsewhere} \end{cases}$$

We have seen before that

$$\phi_{XX}(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

Hence,

$$\begin{aligned} \Phi_{XX}(f) &= \frac{A^2}{2}\mathcal{F}[\cos(2\pi f_c \tau)] \\ &= \frac{A^2}{4}\left(\delta(f - f_c) + \delta(f + f_c)\right) \end{aligned}$$

Properties of $\Phi_{XX}(f)$

1. $\Phi_{XX}(0) = \int_{-\infty}^{\infty} \phi_{XX}(\tau) d\tau$
2. $\int_{0-}^{0+} \Phi_{XX}(f) df =$ dc power
3. $\phi_{XX}(0) = \int_{-\infty}^{\infty} \Phi_{XX}(f) df =$ total power
4. $\Phi_{XX}(f) \geq 0$ for all f. Power is never negative.
5. $\Phi_{XX}(f) = \Phi_{XX}(-f)$ (even function) if $X(t)$ is a real random process.
6. $\Phi_{XX}(f)$ is always real.