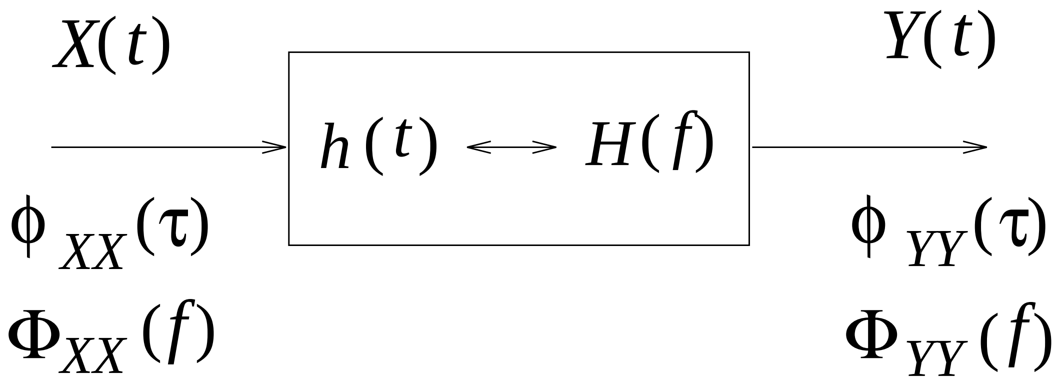
EE4601
Communication Systems

Lecture 7
Linear Systems

Linear Systems



Linear Systems

Suppose that the input to the linear system $h(t)$ is a wide sense stationary random process $X(t)$, with mean μ_X and autocorrelation $\phi_{XX}(\tau)$.

The input and output waveforms are related by the convolution integral

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau \ .$$

Hence,

$$Y(f) = H(f)X(f) \ .$$

The output mean is

$$\mu_Y = \int_{-\infty}^{\infty} h(\tau)\mathrm{E}[X(t-\tau)]d\tau = \mu_X \int_{-\infty}^{\infty} h(\tau)d\tau = \mu_X H(0) \ .$$

This is just the mean (dc component) of the input signal multiplied by the dc gain of the filter.

Linear Systems

The output autocorrelation is

$$\begin{aligned}
\phi_{YY}(\tau) &= \mathrm{E}[Y(t+\tau)Y(t)] \\
&= \mathrm{E}\left[\int_{-\infty}^{\infty} h(\alpha)X(t+\tau-\alpha)\int_{-\infty}^{\infty} h(\beta)X(t-\beta)d\beta d\alpha\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)\mathrm{E}\left[X(t+\tau-\alpha)X(t-\beta)\right]d\beta d\alpha \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)\phi_{XX}(\tau-\alpha+\beta)d\beta d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)\int_{-\infty}^{\infty} h(\beta)\phi_{XX}(\tau+\beta-\alpha)d\alpha d\beta \\
&= \left\{\int_{-\infty}^{\infty} h(\beta)\phi_{XX}(\tau+\beta)d\beta\right\} * h(\tau) \\
&= h(-\tau) * \phi_{XX}(\tau) * h(\tau) \ .
\end{aligned}$$

Taking transforms, the output psd is

$$\begin{aligned}
\Phi_{YY}(f) &= H^*(f)\Phi_{XX}(f)H(f) \\
&= |H(f)|^2\,\Phi_{XX}(f) \ .
\end{aligned}$$

Cross-correlation and Cross-covariance

If $X(t)$ and $Y(t)$ are each wide sense stationary and jointly wide sense stationary, then

$$\begin{aligned}\boldsymbol{\phi}_{XY}(t,t+\tau) &= \mathrm{E}[X(t)Y(t+\tau)] = \boldsymbol{\phi}_{XY}(\tau) \\ \boldsymbol{\mu}_{XY}(t,t+\tau) &= \boldsymbol{\mu}_{XY}(\tau) = \boldsymbol{\phi}_{XY}(\tau) - \mu_x\mu_y\end{aligned}$$

The crosscorrelation function $\phi_{XY}(\tau)$ has the following properties.

1. $\phi_{XY}(\tau) = \phi_{YX}(-\tau)$
2. $|\phi_{XY}(\tau)| \leq \frac{1}{2}[\phi_{XX}(0) + \phi_{YY}(0)]$
3. $|\phi_{XY}(\tau)|^2 \leq \phi_{XX}(0)\phi_{YY}(0)$ if $X(t)$ and $Y(t)$ have zero mean.

Example

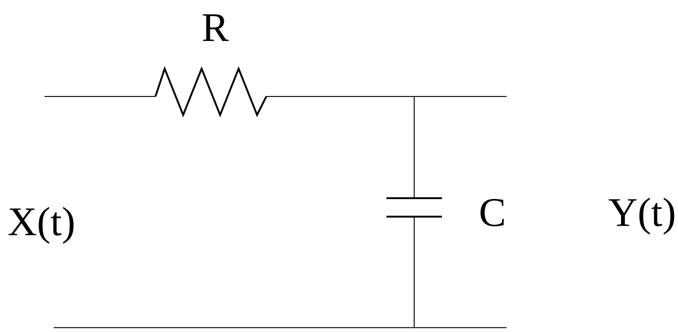
Consider the linear system shown in the previous example. The crosscorrelation between the input process $X(t)$ and the output process $Y(t)$ is

$$\begin{aligned}
\phi_{YX}(\tau) &= \mathrm{E}[Y(t)X(t+\tau)] \\
&= \mathrm{E}\left[\int_{-\infty}^{\infty} h(\alpha)X(t-\alpha)d\alpha X(t+\tau)\right] \\
&= \int_{-\infty}^{\infty} h(\alpha)\mathrm{E}\left[X(t-\alpha)X(t+\tau)\right]d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)\phi_{XX}(\tau+\alpha)d\alpha \\
&= h(-\tau) * \phi_{XX}(\tau)
\end{aligned}$$

The cross power spectral density is

$$S_{YX}(f) = H^*(f)S_{XX}(f)$$

Example



Example

The transfer function of the filter is

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

Suppose that $\phi_{XX}(f) = e^{-\alpha|\tau|}$. What is $\phi_{YY}(\tau)$?

We have

$$S_{YY}(f) = |H(f)|^2 S_{XX}(f)$$

where

$$|H(f)|^2 = \frac{1}{1 + (2\pi fRC)^2}$$

$$S_{XX}(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

Hence,

$$S_{YY}(f) = \frac{1}{1 + (2\pi fRC)^2} \cdot \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

Example

Do you remember *partial fractions*? Now you need them!
We write

$$S_{YY}(f) = \frac{A}{\alpha^2 + (2\pi f)^2} + \frac{B}{1 + (2\pi f RC)^2}$$

and solve for A and B. We have

$$A(1 + (2\pi f RC)^2) + B(\alpha^2 + (2\pi f)^2) = 2\alpha$$

Clearly,

$$A + B\alpha^2 = 2\alpha$$
$$A(2\pi f RC)^2 + B(2\pi f)^2 = 0$$

From the second equation

$$A = -\frac{B}{(RC)^2} = -B\beta^2$$

where $\beta = 1/(RC)$.

Example

Then using the first equation

$$B = \frac{2\alpha}{\alpha^2 - \beta^2}$$

Also,

$$A = -B\beta^2 = -\frac{2\alpha\beta^2}{\alpha^2 - \beta^2}$$

Finally,

$$S_{YY}(f) = \frac{\beta^2}{\beta^2 - \alpha^2} \cdot \frac{2\alpha}{\alpha^2 + (2\pi f)^2} + \frac{\alpha\beta}{\alpha^2 - \beta^2} \cdot \frac{2\beta}{\beta^2 + (2\pi f)^2}$$

Now take inverse Fourier transforms to get

$$\phi_{YY}(\tau) = \frac{\beta^2}{\beta^2 - \alpha^2} \cdot e^{-\alpha|\tau|} + \frac{\alpha\beta}{\alpha^2 - \beta^2} \cdot e^{-\beta|\tau|}$$