EE4601
Communication Systems

Lecture 8
Noise and Matched Filters

Thermal Noise

Thermal noise affect all communication receivers.

From fundamental physics (which we will not go into here) the power spectral density of thermal noise is

$$\Phi_{nn}(f) = \frac{2h|f|}{e^{h|f|/kT} - 1} \text{ watts/Hz}$$

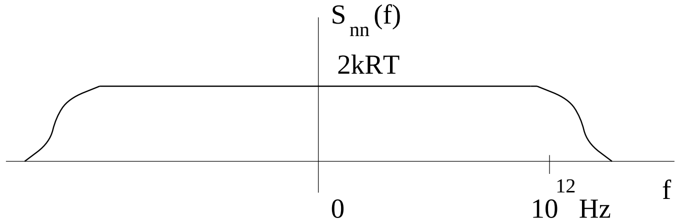
where

$$\begin{aligned} h &= 6.62 \times 10^{-34} \text{ Joules} = \text{Plank's constant} \\ k &= 1.37 \times 10^{-23} \text{ Joules/degree} = \text{Boltzmann's constant} \end{aligned}$$

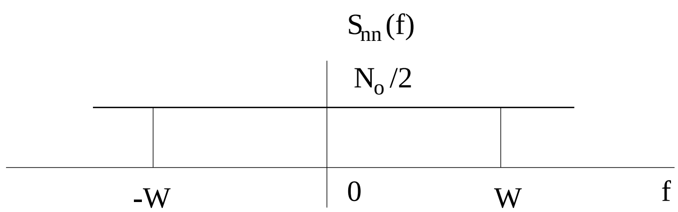
Using the Taylor series expansion $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$ gives

$$\begin{aligned} \Phi_{nn}(f) &\approx \frac{2h|f|}{1 + h|f|/kT - 1} \\ &= 2kT \text{ watts/Hz} \end{aligned}$$

White Noise



Over a narrow bandwidth of frequencies the noise spectral density can be considered "flat"



White Noise

If we assume the bandwidth W is infinite (idealization), then the autocorrelation of the noise is

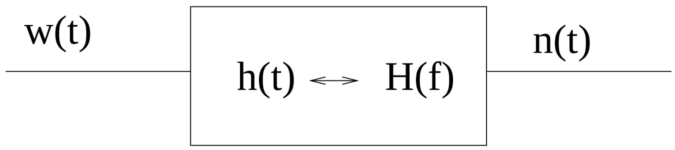
$$\phi_{ww}(\tau) = \mathcal{F}^{-1}\left\{N_o/2\right\} = \frac{N_o}{2}\delta(\tau)$$

where we use a subscript "w" to emphasize that the noise is white.
Note that $n(t)$ is *uncorrelated* with $n(t+\tau)$ for any $\tau \neq 0$.

The noise power in bandwidth W is

$$P_n = 2 \times W \times \frac{N_o}{2} = N_o W \quad \text{watts}$$

Filtered White Noise



If the input noise spectral density is $\Phi_{ww}(f) = N_o/2$, then the output noise spectral density is

$$\Phi_{nn}(f) = \frac{N_o}{2} |H(f)|^2$$

For example, consider the ideal low-pass filter

$$H(f) = \text{rect}\left(\frac{f}{2W}\right)$$

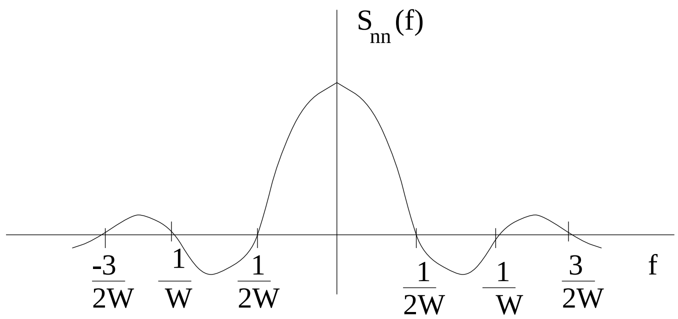
Then

$$\Phi_{nn}(f) = \frac{N_o}{2}\text{rect}\left(\frac{f}{2W}\right)$$

Filtered White Noise

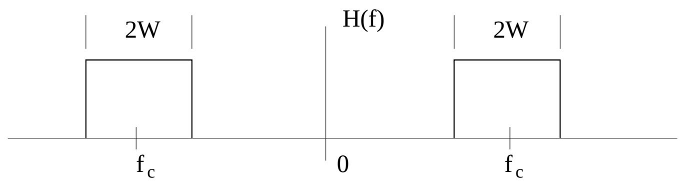
The autocorrelation function of the ideal low-pass filtered noise is

$$\begin{aligned}\phi_{nn}(\tau) &= \frac{N_o}{2} 2W \operatorname{sinc}(2W\tau) \\ &= N_o W \operatorname{sinc}(2W\tau)\end{aligned}$$



Observe that regularly spaced samples of $n(t)$ taken $1/(2W)$ seconds apart are *uncorrelated*. Interesting!

Bandpass Filtered White Noise



$$H(f) = \text{rect}\left(\frac{f - f_c}{2W}\right) + \text{rect}\left(\frac{f + f_c}{2W}\right)$$

$$\begin{aligned}
\Phi_{nn}(f) &= \frac{N_o}{2}\left[\text{rect}\left(\frac{f - f_c}{2W}\right) + \text{rect}\left(\frac{f + f_c}{2W}\right)\right] \\
\phi_{nn}(\tau) &= \frac{N_o}{2} \cdot 2W \text{sinc}(2W\tau) \cdot 2\cos 2\pi f_c \tau \\
&= 2N_o W \text{sinc}(2W\tau) \cdot \cos 2\pi f_c \tau
\end{aligned}$$

Noise Equivalent Bandwidth

Consider an arbitatry filter with tranfer function $H(f)$. If the input to the filter is white noise with power spectral density $N_o/2$, then the noise power at the output of the filter is

$$\begin{aligned} N_{\text{out}} &= \frac{N_o}{2}\int_{-\infty}^{\infty} |H(f)|^2 df \\ &= N_o \int_0^{\infty} |H(f)|^2 df \end{aligned}$$

Next suppose that the same noise process is applied to an ideal low-pass filter with bandwidth B and zero frequency response $H(0)$. The noise at the output of the filter is

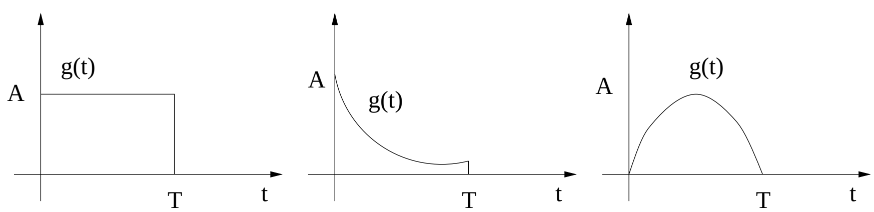
$$N_{\text{out}} = N_o B H^2(0)$$

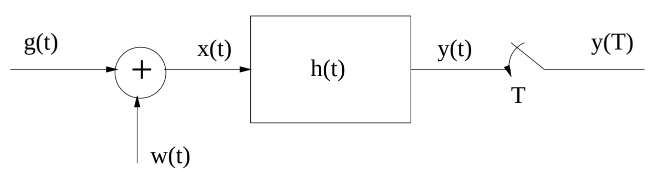
Equating the above two equations give the *noise equivalent bandwdith*

$$B = \frac{\int_0^{\infty} |H(f)|^2 df}{H^2(0)}$$

Basic Problem

A pulse $g(t)$ is transmitted over a noisy channel, representing a "0" or "1". The pulse is assumed to have duration T.





$\Phi_{ww}(f) = \frac{N_o}{2}$

Given the knowledge of $g(t)$, how do we choose $h(t)$ to minimize the effects of noise?

Matched Filter

$$y(t) = g_o(t) + n(t)$$

where

$$\begin{aligned} g_o(t) &= g(t) * h(t) \\ n(t) &= w(t) * h(t) \end{aligned}$$

We wish to maximize the <u>peak pulse signal-to-noise ratio</u>

$$\eta = \frac{|g_o(t)|^2}{E[n^2(t)]} = \frac{\text{instantaneous signal power}}{\text{average noise power}}$$

where T = sampling instant We have $\Phi_{nn}(f) = |H(f)|^2 \Phi_{ww}(f) = \frac{N_o}{2}|H(f)|^2$

$$\begin{aligned} E[n^2(t)] &= \phi_{nn}(0) = \int_{-\infty}^{\infty} \Phi_{nn}(f) df = \frac{N_o}{2} \int_{-\infty}^{\infty} |H(f)|^2 df \\ g_o(T) &= \int_{-\infty}^{\infty} G(f) H(f) e^{j2\pi fT} df \end{aligned}$$

Matched Filter

Then,

$$\eta = \frac{|\int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT}df|^2}{\frac{N_o}{2}\int_{-\infty}^{\infty}|H(f)|^2 df}$$

Choose $H(f)$ to maximize η.
Apply the Schwartz inequality (Read Appendix 5)

$$|\int_{-\infty}^{\infty} x(f)y(f)|^2 \leq \int_{-\infty}^{\infty}|x(f)|^2 df \int_{-\infty}^{\infty}|y(f)|^2 df$$

with equality iff $x(f) = ky^*(f)$, k - arbitrary scalar constant
Hence,

$$\left|\int_{-\infty}^{\infty} G(f)H(f)e^{j2\pi fT}df\right|^2 \leq \int_{-\infty}^{\infty}|G(f)|^2 df \int_{-\infty}^{\infty}|H(f)|^2 df$$

and

$$\eta \leq \frac{2}{N_o}\int_{-\infty}^{\infty}|G(f)|^2 df$$

Since the RHS does not depend on $H(f)$, we maximize η by choosing

$$H_{\text{opt}}(f) = kG^*(f)e^{-j2\pi fT} \leftrightarrow kg(T-t) = h_{\text{opt}}(t)$$

Matched Filter

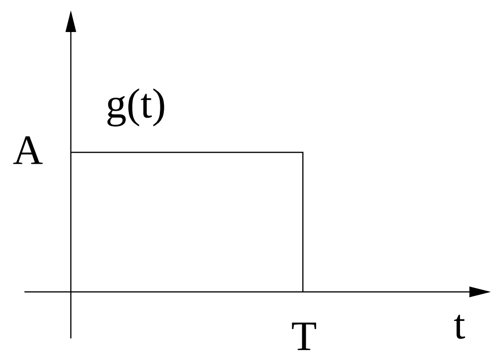
This gives

$$\eta_{\max} = \frac{2}{N_o}\int_{-\infty}^{\infty}|G(f)|^2 df = \frac{E}{N_o/2}$$

Recall Rayleigh's energy theorem

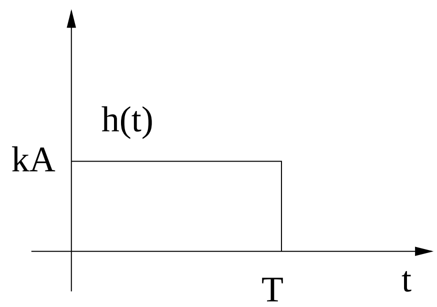
$$E = \int_{-\infty}^{\infty}|g(t)|^2 dt = \int_{-\infty}^{\infty}|G(f)|^2 df$$

Example: $g(t) = AU_T(t) = AU(t) - AU(t-T)$

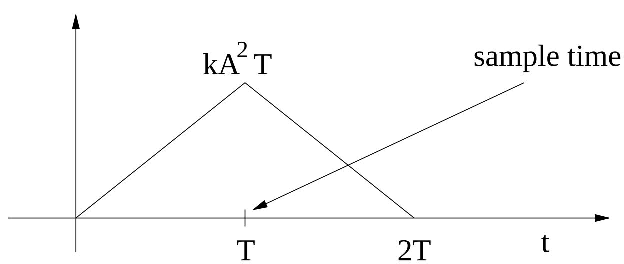


Matched Filter

$$h(t) = kg(T - t)$$



$$g_o(t) = g(t) * h(t)$$



[0]©2007, Georgia Institute of Technology (lect8_13)