EE6604
Personal & Mobile Communications

Week 14

MIMO Channels

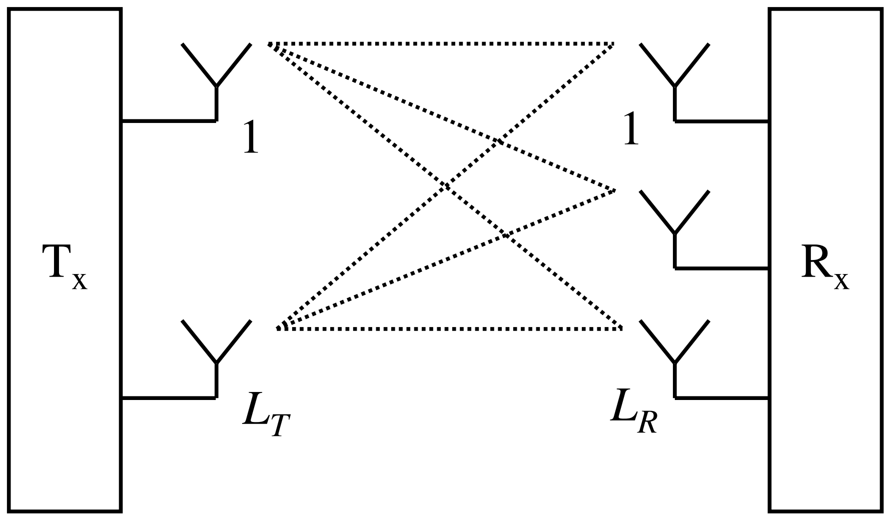
Alamouti Space-time Coding

Spatial Multiplexing

Reading: 6.10, 6.11, 6.12

MIMO Channels

- A multiple-input multiple-output (MIMO) system is one that consists of multiple transmit and receive antennas.



MIMO system with multiple transmit and multiple receiver antennas.

MIMO Channels

- For a system consisting of L_t transmit and L_r receive antennas, the channel can be described by $L_t \times L_r$ matrix

$$\mathbf{G}(t,\tau) = \begin{bmatrix} g_{1,1}(t,\tau) & g_{1,2}(t,\tau) & \cdots & g_{1,L_t}(t,\tau) \\ g_{2,1}(t,\tau) & g_{2,2}(t,\tau) & \cdots & g_{2,L_t}(t,\tau) \\ \vdots & \vdots & & \vdots \\ g_{L_r,1}(t,\tau) & g_{L_r,2}(t,\tau) & \cdots & g_{L_r,L_t}(t,\tau) \end{bmatrix} ,$$

  - $g_{qp}(t,\tau)$ denotes the time-varying sub-channel impulse response between the pth transmitter antenna and qth receiver antenna.

- Suppose that the complex envelopes of the signals transmitted from the L_t transmit antennas are:

$$\tilde{\mathbf{s}}(t) = (\tilde{s}_1(t), \tilde{s}_2(t), \ldots, \tilde{s}_{L_t}(t))^{\mathrm{T}} ,$$

  where $\tilde{s}_p(t)$ is the signal transmitted from the pth transmit antenna.

- Let

$$\tilde{\mathbf{r}}(t) = (\tilde{r}_1(t), \tilde{r}_2(t), \ldots, \tilde{r}_{L_r}(t))^{\mathrm{T}} ,$$

  denote the vector of received complex envelopes, where $\tilde{r}_q(t)$ is the signal received at the qth receiver antenna. Then

$$\tilde{\mathbf{r}}(t) = \int_0^t \mathbf{G}(t,\tau)\tilde{\mathbf{s}}(t-\tau)d\tau$$

MIMO Channels - Special Cases

- Under conditions of flat fading

$$\mathbf{G}(t,\tau) = \mathbf{G}(t)\delta(\tau - \hat{\tau}) \ ,$$

  where $\hat{\tau}$ is the delay through the channel and

$$\tilde{\mathbf{r}}(t) = \mathbf{G}(t)\tilde{\mathbf{s}}(t - \hat{\tau}) \ .$$

- If the MIMO channel is characterized by slow fading, then

$$\tilde{\mathbf{r}}(t) = \int_0^t \mathbf{G}(\tau)\tilde{\mathbf{s}}(t - \tau)d\tau \ .$$

  - In this case, the channel matrix $\mathbf{G}(\tau)$ remains constant over the duration of the transmitted waveform $\tilde{\mathbf{s}}(t)$, but can vary from one channel use to the next, where a channel use may be defined as the transmission of either a single modulated symbol or a vector of modulated symbols.
  - Sometimes this is called a randomly static channel or a block fading channel.

- Finally, if the MIMO channel is characterized by slow flat fading, then

$$\tilde{\mathbf{r}}(t) = \mathbf{G}\tilde{\mathbf{s}}(t) \ .$$

MIMO Channel Models - Classification

- MIMO channel models can be classified as either *physical* or *analytical* models.
- The analytical models characterize the MIMO sub-channel impulse responses in a mathematical manner without explicitly considering the underlying electromagnetic wave propagation.
  - Analytical MIMO channel models are most often used under slowly and flat fading conditions.
  - The various analytical models generate the MIMO matrices as realizations of complex Gaussian random variables having specified means and correlations.
  - To model Rician fading, the channel matrix can be divided into a deterministic part and a random part, i.e.,

$$\mathbf{G} = \sqrt{\frac{K}{K+1}}\bar{\mathbf{G}} + \sqrt{\frac{1}{K+1}}\mathbf{G}_s$$

    where $\mathrm{E}[\mathbf{G}] = \sqrt{\frac{K}{K+1}}\bar{\mathbf{G}}$ is the LoS or specular component and $\sqrt{\frac{K}{K+1}}\mathbf{G}_s$. is the scatter component assumed to have zero-mean.
  - To simply our further characterization of the MIMO channel, assume for the time being that $K = 0$, so that $\mathbf{G} = \mathbf{G}_s$.

i.i.d. MIMO Channel Model

- The simplest MIMO model assumes that the entries of the matrix $\mathbf{G}$ are independent and identically distributed (i.i.d) complex Gaussian random variables.
  - This model corresponds to the so called "rich scattering" or spatially white environment.
  - Such an independence assumption simplifies the performance analysis of various digital signaling schemes operating on MIMO channels.
  - In reality the sub-channels will be correlated and, therefore, the i.i.d. model will lead to optimistic performance estimates.
  - A variety of more sophisticate models have been introduced to account for spatial correlation of the sub-channels.

Correlated MIMO Channel Models

- Consider the vector $\mathbf{g} = \text{vec}\{\mathbf{G}\}$ where

$$\mathbf{G} = [\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{L_t}] \ , \quad \mathbf{g}_j = (g_{1,j}, g_{2,j}, \ldots, g_{L_r,j})^{\mathrm{T}}$$

and

$$\text{vec}\{\mathbf{G}\} = [\mathbf{g}_1^{\mathrm{T}}, \mathbf{g}_2^{\mathrm{T}}, \ldots, \mathbf{g}_{L_t}^{\mathrm{T}}]^{\mathrm{T}}.$$

- The vector $\mathbf{g}$ is a column vector of length $n = L_t L_r$. The vector $\mathbf{g}$ is zero-mean complex Gaussian random vector and its statistics are fully specified by the $n \times n$ covariance matrix $\mathbf{R}_G = \mathrm{E}[\mathbf{g}\mathbf{g}^{\mathrm{H}}]$, where $\mathbf{g}^{\mathrm{H}}$ is the complex conjugate transpose of $\mathbf{g}$.

- Hence, $\mathbf{g} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}_G)$ and, if $\mathbf{R}_G$ is invertible, the probability density function of $\mathbf{g}$ is

$$p(\mathbf{g}) = \frac{1}{(2\pi)^n \text{det}(\mathbf{R}_G)} e^{-\frac{1}{2}\mathbf{g}^{\mathrm{H}}\mathbf{R}_G^{-1}\mathbf{g}} \ , \quad \mathbf{g} \in \mathcal{C}^n \ .$$

- Realizations of the MIMO channel with the above distribution can be generated by

$$\mathbf{G} = \text{unvec}(\mathbf{g}) \quad \text{with} \quad \mathbf{g} = \mathbf{R}_G^{1/2}\mathbf{w} \ .$$

Here, $\mathbf{R}_G^{1/2}$ is any matrix square root of $\mathbf{R}_G$, i.e., $\mathbf{R}_G = \mathbf{R}_G^{1/2}(\mathbf{R}_G^{1/2})^{\mathrm{H}}$, and $\mathbf{w}$ is a length n vector where $\mathbf{w} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$.

Correlated MIMO Channel Models

- To find the square root of the matrix $\mathbf{R}_G$, we can use eigenvalue decomposition.
- Note that the matrix $\mathbf{R}_G$ is Hermitian, i.e., $\mathbf{R}_G = \mathbf{R}_G^{\mathrm{H}}$.
- It follows that $\mathbf{R}_G$ has the eigenvalue decomposition $\mathbf{R}_G = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{H}}$, where $\mathbf{U}$ is a unitary matrix, i.e., $\mathbf{U}\mathbf{U}^{\mathrm{H}} = \mathbf{I}$.
- Then we have $\mathbf{R}_G^{1/2} = \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{U}^{\mathrm{H}}$
- To verify this solution, we note that

$$\begin{aligned}\mathbf{R}_G &= \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{U}^{\mathrm{H}}\mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{U}^{\mathrm{H}} \\ &= \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}\mathbf{U}^{\mathrm{H}} \\ &= \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{H}}\end{aligned}$$

- To find the matrix $\mathbf{U}$, we formulate

$$\mathbf{R}_G\mathbf{u} = \lambda\mathbf{u}$$

  and solve for λ and $\mathbf{u}$. This can be done by solving the N (assuming matrix $\mathbf{R}_G$ is full rank) roots of the polynomial

$$p(\lambda) = \det(\mathbf{R}_G - \lambda\mathbf{I}) = 0$$

- For each solution λ_i, we have the specific eigenvalue equation which we solve for $\mathbf{u}$

$$(\mathbf{R}_G - \lambda\mathbf{I})\mathbf{u} = \mathbf{0}$$

Kronecker Model

- The Kronecker model assumes that the spatial correlation at the transmitter and receiver is separable.
- This is equivalent to restricting the correlation matrix $\mathbf{R}_H$ to have the Kronecker product form

$$\mathbf{R}_G = \mathbf{R}_T \otimes \mathbf{R}_R$$

where

$$\mathbf{R}_T = \mathrm{E}[\mathbf{G}^{\mathrm{H}}\mathbf{G}] \qquad \mathbf{R}_R = \mathrm{E}[\mathbf{G}\mathbf{G}^{\mathrm{H}}] \ .$$

are the $L_t \times L_t$ and $L_r \times L_r$ transmit and receive correlation matrices respectively, and $\otimes$ is the "Kronecker product."

  - For example, the Kronecker product of an $n \times n$ matrix $\mathbf{A}$ and an $m \times m$ matrix $\mathbf{B}$ would be

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{n1}\mathbf{B} & \cdots & a_{nn}\mathbf{B} \end{bmatrix} \ .$$

- Under the above Kronecker assumption,

$$\mathbf{g} = (\mathbf{R}_T \otimes \mathbf{R}_R)^{1/2}\,\mathbf{w}$$

and

$$\mathbf{G} = \mathbf{R}_R^{1/2}\mathbf{W}\mathbf{R}_T^{1/2} \ ,$$

where $\mathbf{W}$ is an $L_r \times L_t$ matrix consisting of i.i.d. zero mean complex Gaussian random variables.

Kronecker Model

- If the elements of $\mathbf{G}$ could be arbitrarily selected, then the correlation functions would be a function of four sub-channel index parameters, i.e.,

$$\mathrm{E}[g_{qp} g^*_{\tilde{q}\tilde{p}}] = \phi(q, p, \tilde{q}, \tilde{p})$$

where g_{qp} is the channel between the pth transmit and qth receive antenna.

- However, due to the Kronecker property, $\mathbf{R}_G = \mathbf{R}_T \otimes \mathbf{R}_R$, the elements of $\mathbf{G}$ are structured.

- One implication of the Kronecker property is "spatial" stationarity

$$\mathrm{E}[g_{qp} g^*_{\tilde{q}\tilde{p}}] = \phi(q - \tilde{q}, p - \tilde{p}) \ ,$$

which implies that the sub-channel correlations are determined not by their position in the matrix $\mathbf{G}$, but by their positional difference.

- In addition, to the stationary property, manipulation of the Kronecker product form in $\mathbf{R}_G = \mathbf{R}_T \otimes \mathbf{R}_R$ implies that

$$\mathrm{E}[g_{qp} g^*_{\tilde{q}\tilde{p}}] = \phi(q - \tilde{q}, p - \tilde{p}) = \phi_R(q - \tilde{q}) \cdot \phi_T(p - \tilde{p}) \ ,$$

meaning that the correlation can be separated into two parts: a transmitter part and a receiver part, and both parts are stationary.

- Finally, it can be shown that the Kronecker property, $\mathbf{R}_G = \mathbf{R}_T \otimes \mathbf{R}_R$, holds if and only if the above separable property holds.

Weichselberger Model

- The Weichselberger model overcomes the separable requirement of the channel correlation functions of the Kronecker model.

- Consider the eigenvalue decomposition of the transmitter and receiver correlation matrices,

$$\begin{aligned} \mathbf{R}_T &= \mathbf{U}_T \mathbf{\Lambda}_T \mathbf{U}_T^{\mathrm{H}} \\ \mathbf{R}_R &= \mathbf{U}_R \mathbf{\Lambda}_R \mathbf{U}_R^{\mathrm{H}} \end{aligned}$$

  - $\mathbf{\Lambda}_T$ and $\mathbf{\Lambda}_R$ are diagonal matrices containing the eigenvalues of, and $\mathbf{U}_T$ and $\mathbf{U}_R$ are unity matrices containing the eigenvectors of, $\mathbf{R}_T$ and $\mathbf{R}_R$.

- The Weichselberger model constructs the matrix $\mathbf{G}$ as

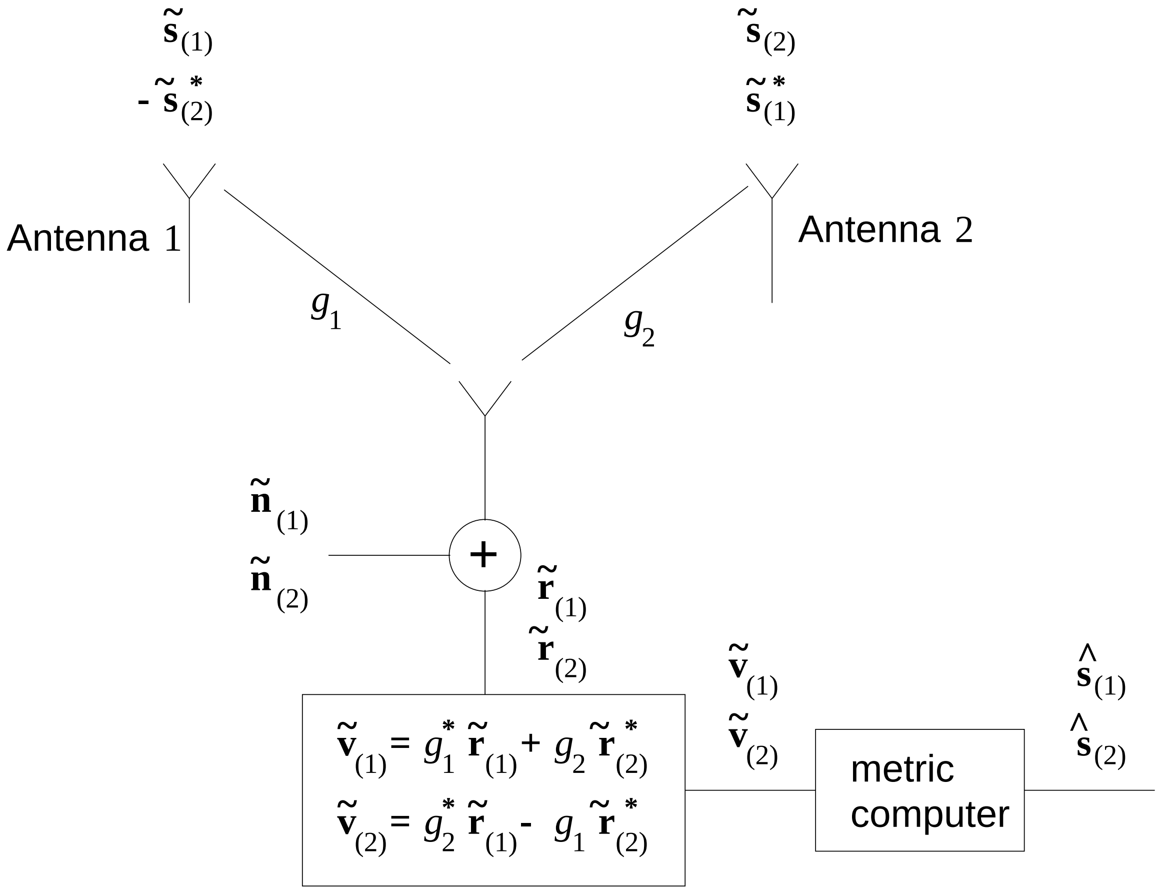
$$\mathbf{G} = \mathbf{U}_R \left(\tilde{\mathbf{\Omega}} \odot \mathbf{W} \right) \mathbf{U}_T^{\mathrm{T}} \ ,$$

  where $\mathbf{W}$ is an $L_r \times L_t$ matrix consisting of i.i.d. zero mean complex Gaussian random variables and $\odot$ denotes the Schur-Hadamard product (element-wise matrix multiplication), and $\mathbf{\Omega}$ is an $L_r \times L_t$ coupling matrix whose non-negative real values determine the average power coupling between the transmitter and receiver eigenvectors. The matrix $\tilde{\mathbf{\Omega}}$ is the element-wise square root of $\mathbf{\Omega}$.

- The Kronecker model is a special case of the Weichselberger model obtained with the coupling matrix $\mathbf{\Omega} = \boldsymbol{\lambda}_R \boldsymbol{\lambda}_T^{\mathrm{T}}$, where $\boldsymbol{\lambda}_R$ and $\boldsymbol{\lambda}_T$ are column vectors containing the eigenvalues of $\mathbf{\Lambda}_T$ and $\mathbf{\Lambda}_R$, respectively.

Transmit Diversity - Almouti Scheme

- **Transmitter diversity** uses multiple transmit antennas to provide the receiver with multiple uncorrelated replicas of the same signal.
- The complexity of having multiple antenna is placed on the transmitter which may be shared among many receivers.
- Transmit diversity schemes require three functions:
  - encoding and transmission of the information sequence at the transmitter
  - combining scheme at the receiver
  - decision rule for making decisions
- We consider a simple repetition transmit diversity scheme with maximum likelihood combining at the receiver. This is the Alamouti transmit diversity scheme.



Space-time diversity receiver for 2×1 *diversity.*

- The received complex vectors are

$$\begin{aligned} \tilde{\mathbf{r}}_{(1)} &= g_1 \tilde{\mathbf{s}}_{(1)} + g_2 \tilde{\mathbf{s}}_{(2)} + \tilde{\mathbf{n}}_{(1)} \\ \tilde{\mathbf{r}}_{(2)} &= -g_1 \tilde{\mathbf{s}}^*_{(2)} + g_2 \tilde{\mathbf{s}}^*_{(1)} + \tilde{\mathbf{n}}_{(2)} \end{aligned}$$

  $\tilde{\mathbf{r}}_{(1)}$ and $\tilde{\mathbf{r}}_{(2)}$ represent the received vectors at time t and $t+T$, respectively, and $\tilde{\mathbf{n}}_{(1)}$ and $\tilde{\mathbf{n}}_{(2)}$ are the corresponding noise vectors.

- The combiner constructs the following two signal vectors

$$\begin{aligned} \tilde{\mathbf{v}}_{(1)} &= g_1^* \tilde{\mathbf{r}}_{(1)} + g_2 \tilde{\mathbf{r}}^*_{(2)} \\ \tilde{\mathbf{v}}_{(2)} &= g_2^* \tilde{\mathbf{r}}_{(1)} - g_1 \tilde{\mathbf{r}}^*_{(2)} \end{aligned}$$

  Afterwards, the receiver applies the vectors $\tilde{\mathbf{v}}_{(1)}$ and $\tilde{\mathbf{v}}_{(2)}$ in a sequential fashion to the metric computer, to make decisions on the symbols $\tilde{\mathbf{s}}_{(1)}$ and $\tilde{\mathbf{s}}_{(2)}$ by maximizing the two respective decision variables

$$\begin{aligned} \mu(\tilde{\mathbf{s}}_{(1),m}) &= \mathrm{Re}\left\{\tilde{\mathbf{v}}_{(1)} \cdot \tilde{\mathbf{s}}^*_{(1),\mathrm{m}}\right\} - E_m(|g_1|^2 + |g_2|^2) \\ \mu(\tilde{\mathbf{s}}_{(2),m}) &= \mathrm{Re}\left\{\tilde{\mathbf{v}}_{(2)} \cdot \tilde{\mathbf{s}}^*_{(2),\mathrm{m}}\right\} - E_m(|g_1|^2 + |g_2|^2) \end{aligned}$$

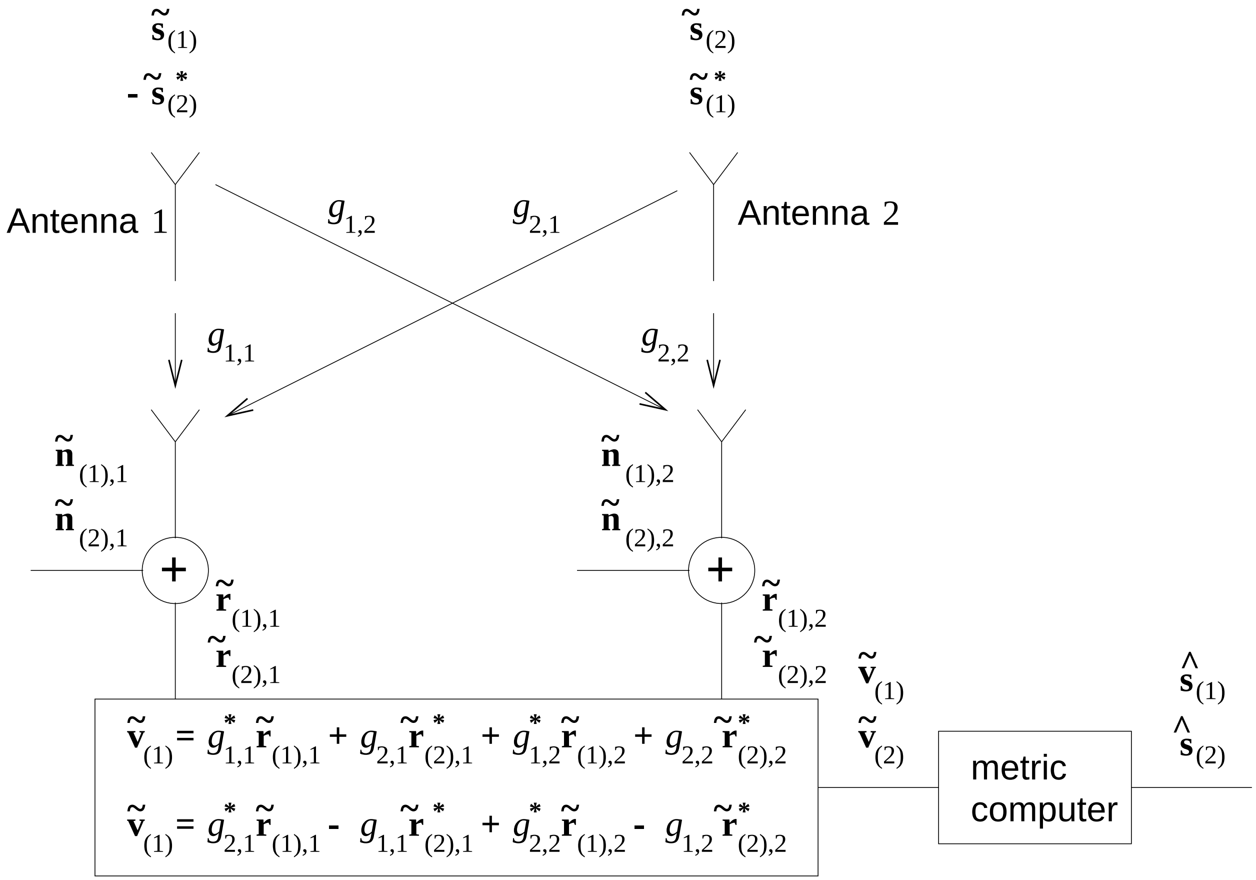
- We have

$$\begin{aligned}\tilde{\mathbf{v}}_{(1)} &= (\alpha_1^2 + \alpha_2^2)\tilde{\mathbf{s}}_{(1)} + g_1^*\tilde{\mathbf{n}}_{(1)} + g_2\tilde{\mathbf{n}}_{(2)}^* \\ \tilde{\mathbf{v}}_{(2)} &= (\alpha_1^2 + \alpha_2^2)\tilde{\mathbf{s}}_{(2)} - g_1\tilde{\mathbf{n}}_{(2)}^* + g_2^*\tilde{\mathbf{n}}_{(1)}\end{aligned}$$

- Compare with MRC
  - With 1×2 diversity and MRC

$$\begin{aligned}\tilde{\mathbf{r}} &= g_1^*\tilde{\mathbf{r}}_1 + g_2^*\tilde{\mathbf{r}}_2 \\ &= (\alpha_1^2 + \alpha_2^2)\tilde{\mathbf{s}}_m + g_1^*\tilde{\mathbf{n}}_1 + g_2^*\tilde{\mathbf{n}}_2\end{aligned}$$

  - The combined signals in each case are the same. The only difference is the phase rotations of the noise vectors which will not change the error probability.
  - However, with transmit diversity the transmit power must be split between two transmit antennas. Hence, 2×1 diversity is 3 dB less power efficient than 1×2 diversity.



Space-time diversity receiver for 2×2 diversity.

Spatial Multiplexing

- In certain types of wireless systems, particularly those using time division duplexing (TDD), knowledge of the channel $\mathbf{G}$ is available at both the transmitter and receiver. In this case, as singular value decomposition (SVD) of the channel matrix $\mathbf{G}$ may be performed.
- Suppose that the channel matrix $\mathbf{G}$ has rank r which is at most $\min\{L_T, L_R\}$. Then

$$\mathbf{G} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^H$$

  where $\mathbf{U}$ is an $L_R \times r$ matrix, $\mathbf{V}$ is an $L_T \times r$ matrix, and $\mathbf{\Lambda}$ is an $r \times r$ diagonal matrix, such that the diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the singular values of the channel matrix $\mathbf{G}$.
- The matrices $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices, meaning that $\mathbf{U}\mathbf{U}^H = \mathbf{I}_{r\times r}$ and $\mathbf{V}\mathbf{V}^H = \mathbf{I}_{r\times r}$ where $\mathbf{I}_{r\times r}$ is the $r \times r$ identity matrix.

Spatial Multiplexing (cont'd)

- Given knowledge that the channel matrix $\mathbf{G}$ has rank r at the transmitter, r symbols are sent over the channel. The $r \times 1$ transmitted signal vector $\tilde{\mathbf{s}}$ is precoded at the transmitter by using the linear transformation
$$\tilde{\mathbf{s}}_p = \mathbf{V}\tilde{\mathbf{s}}$$
and transmitted from the L_T transmit antennas.

- The corresponding received signal vector across the L_R receiver antennas is
$$\tilde{\mathbf{r}} = \mathbf{G}\tilde{\mathbf{s}}_p + \tilde{\mathbf{n}} = \mathbf{G}\mathbf{V}\tilde{\mathbf{s}} + \tilde{\mathbf{n}} \ .$$

- At the receiver, the received signal vector $\tilde{\mathbf{r}}$ is processed by the linear transformation $\mathbf{U}^H$ as follows:
$$\begin{aligned} \hat{\mathbf{s}} &= \mathbf{U}^H\tilde{\mathbf{r}} \\ &= \mathbf{U}^H\mathbf{G}\mathbf{V}\tilde{\mathbf{s}} + \mathbf{U}^H\tilde{\mathbf{n}} \\ &= \mathbf{U}^H\mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^H\mathbf{V}\tilde{\mathbf{s}} + \mathbf{U}^H\tilde{\mathbf{n}} \\ &= \boldsymbol{\Lambda}\tilde{\mathbf{s}} + \mathbf{U}^H\tilde{\mathbf{n}} \ . \end{aligned}$$

- Multiplication of the noise vector $\tilde{\mathbf{n}}$ by the unitary matrix $\mathbf{U}^H$ does not alter the statistics of the noise vector. Due to the multiplication of each transmitted symbol $\tilde{s}_k$ by the corresponding singular value λ_k, the r data streams will have different received bit energy-to-noise ratios depending on the particular channel realization.