ECE 6604 Homework #2 Solutions.

$$g(t,\tau) = u_T(\tau)\sin(\Omega t + \phi_0), \quad 0 \le \tau \le T$$

$$T = 10\,\mu s, \quad \Omega = 10\pi$$

a) $T(f,t) = \mathcal{F}\{g(t,\tau)\}$

$$= \sin(\Omega t + \phi_0)\,\mathcal{F}\{u_T(\tau)\}$$

$$= T\,\mathrm{sinc}(\pi f T)\,e^{j\pi f T}\sin(\Omega t + \phi_0)$$

b) $\tilde{r}(t) = g(t,\tau) * \tilde{s}(t)$

$$= \int_0^t g(t,\tau)\,\tilde{s}(t-\tau)\,d\tau$$

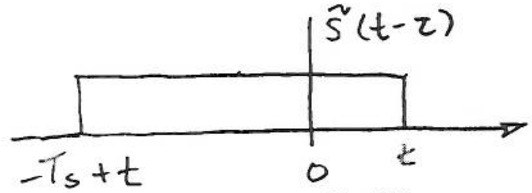
$$= \int_0^t u_T(\tau)\sin(\Omega t + \phi_0)\,\tilde{s}(t-\tau)\,d\tau$$

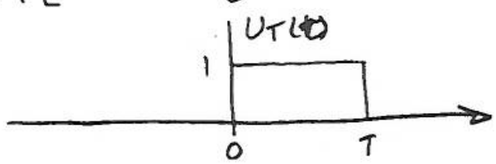
where $u_T(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{else} \end{cases}$

$$\tilde{s}(t) = \begin{cases} 1, & 0 \le t \le T_s \\ 0, & \text{else} \end{cases}$$

This problem is easiest to solve by using graphical convolution. There are actually 2 cases to consider 1. $T < T_s$ and 2. $T > T_s$.

1. Consider the case $T < T_s$.





i) For $t \le 0$, $\tilde{r}(t) = 0$

ii) For $0 \le t \le T$, $\tilde{r}(t) = \{\sin(\Omega t + \phi_0)\} \int_0^t d\tau$
$= \{\sin(\Omega t + \phi_0)\} t$

iii) For $T \le t \le T_s$, $\tilde{r}(t) = T \sin(\Omega t + \phi_0)$

iv) For $T_s \le t \le T + T_s$, $\tilde{r}(t) = (T + T_s - t) \sin(\Omega t + \phi_0)$

v) For $t > T + T_s$, $\tilde{r}(t) = 0$

Summary

$$\tilde{r}(t) = \begin{cases} 0, & t \le 0 \\ t \sin(\Omega t + \phi_0), & 0 \le t \le T \\ T \sin(\Omega t + \phi_0), & T \le t \le T_s \\ (T + T_s - t) \sin(\Omega t + \phi_0), & T_s \le t \le T + T_s \\ 0, & t > T + T_s \end{cases}$$

2. The case $T > T_s$ is similar, but results in the following

$$\tilde{r}(t) = \begin{cases} 0, & t \le 0 \\ t \sin(\Omega t + \phi_0), & 0 \le t \le T_s \\ T_s \sin(\Omega t + \phi_0), & T_s \le t \le T \\ (T + T_s - t) \sin(\Omega t + \phi_0), & T \le t \le T + T_s \\ 0, & t > T + T_s \end{cases}$$

2.

$$\phi_g(\tau) = \begin{cases} \frac{1}{2}[1 + \cos 2\pi\tau/T], & 0 \le \tau \le T/2 \\ 0 & \text{elsewhere} \end{cases}$$

a)

$$\phi_T(\Delta f) = \mathcal{F}\{\phi_g(\tau)\}$$

$$= \int_{-\infty}^{\infty} \phi_g(\tau) e^{-j2\pi\Delta f\tau} d\tau$$

$$= \frac{1}{2}\int_0^{T/2} (1 + \cos 2\pi\tau/T) e^{-j2\pi\Delta f\tau} d\tau$$

$$= \frac{1}{2}\int_0^{T/2} \cos 2\pi\Delta f\tau d\tau - \frac{j}{2}\int_0^{T/2} \cos 2\pi\frac{\tau}{T} \sin 2\pi\Delta f\tau\, d\tau$$

$$= \frac{\sin 2\pi\Delta f\tau}{4\pi\Delta f}\Big|_0^{T/2} + \frac{j}{2}\frac{\cos(2\pi\Delta f - 2\pi/T)\tau}{2\cdot(2\pi\Delta f - 2\pi/T)}\Big|_0^{T/2} + \frac{j}{2}\frac{\cos(2\pi\Delta f + 2\pi/T)\tau}{2(2\pi\Delta f + 2\pi/T)}\Big|_0^{T/2}$$

$$= \frac{\sin \pi\Delta f T}{4\pi\Delta f} + \frac{j}{8}\left[\frac{\cos(\pi\Delta f T - \pi) - 1}{\pi\Delta f - \pi/T}\right] + \frac{j}{8}\left[\frac{\cos(\pi\Delta f T + \pi) - 1}{\pi\Delta f + \pi/T}\right]$$

$$= \frac{\sin \pi \Delta f T}{4\pi \Delta f} + \frac{j}{8}\left[\frac{-\cos \pi \Delta f T - 1}{\pi \Delta f - \pi/T}\right] + \frac{j}{8}\left[\frac{-\cos \pi \Delta f T - 1}{\pi \Delta f + \pi/T}\right]$$

$$= \frac{\sin \pi \Delta f T}{4\pi \Delta f} - \frac{j}{8}(\cos \pi \Delta f T + 1)\left[\frac{1}{\pi \Delta f - \pi/T} + \frac{1}{\pi \Delta f + \pi/T}\right]$$

$$= \frac{\sin \pi \Delta f T}{4\pi \Delta f} - \frac{j(\cos \pi \Delta f T + 1)\, 2\pi \Delta f}{8\left((\pi \Delta f)^2 - (\pi/T)^2\right)}$$

b) *Mean delay*

$$\frac{1}{2}\int_0^{T/2} (1 + \cos 2\pi \tau/T)\, d\tau = \left.\frac{\tau}{2} + \frac{1}{8\pi/T}\sin(4\pi \tau/T)\right|_0^{T/2}$$

$$= \frac{T}{4}$$

$$\frac{1}{2}\int_0^{T/2} (\tau + \tau \cos 2\pi \tau/T)\, d\tau = \frac{\tau^2}{4} + \frac{1}{(2\pi/T)^2}\cos(2\pi \tau/T) + \left.\frac{\tau}{2\pi/T}\sin(2\pi \tau/T)\right|_0^{T/2}$$

$$= \frac{T^2}{16} + \frac{1}{(2\pi/T)^2}\left[-1 - 1\right]$$

$$= \frac{T^2}{16} - \frac{T^2}{2\pi^2}$$

$$\mu_\tau = \frac{T}{4} - \frac{2T}{\pi^2} = \left(\frac{1}{4} - \frac{2}{\pi^2}\right)T = .047T$$

$$\frac{1}{2}\int_0^{T/2}\left(\tau^2 + \tau^2\cos 2\pi\tau/T\right)d\tau$$

$$= \left.\frac{\tau^3}{6} + \frac{\tau\cos 2\pi\tau/T}{(2\pi/T)^2} + \frac{(2\pi/T)^2\tau^2 - 2}{2\,(2\pi/T)^3} \times \sin 2\pi\tau/T\right|_0^{T/2}$$

$$= \frac{T^3}{48} - \frac{T/2}{(2\pi/T)^2} = \frac{T^3}{48} - \frac{T^3}{8\pi^2}$$

$$= \left(\frac{1}{48} - \frac{1}{8\pi^2}\right)T^3$$

Hence

$$\sigma_\tau^2 = \frac{\left(\frac{1}{48} - \frac{1}{8\pi^2}\right)\frac{T^3}{4} - \left(\frac{1}{16} - \frac{1}{2\pi^2}\right)^2 T^4}{T/4}$$

$$= \left(\frac{1}{12} - \frac{1}{2\pi^2}\right)T^2 - \left(\frac{1}{8} - \frac{1}{\pi^2}\right)^2 T^3$$

$$= .03267T^2 - .000561T^3$$

c) $\Delta f_c \approx \frac{1}{\sigma_\tau}$ with σ_τ above

d) Suppose $T = 0.1\,ms$

Then $\sigma_\tau = 18.07 \times 10^{-6}$

For GSM $T_s \approx 3.3 \times 10^{-6}$

Hence, the channel is frequency selective.

3. a) $\phi_{gg}(\tau) = \phi_{g_I g_I}(\tau) + j\phi_{g_I g_Q}(\tau)$

$= \phi_{g_I g_I}(\tau)$, since $\phi_{g_I g_Q}(\tau) = 0$

$S_{gg}(f) = S_{g_I g_I}(f)$

$= \frac{\Omega_p}{2} |H(f)|^2$

$= \frac{\Omega_p}{2} \cdot \frac{A^2}{1+(2\pi f\beta)^2}$

$\phi_{gg}(\tau) = \mathcal{F}^{-1}\{S_{gg}(f)\}$

using the transform pair

$$e^{-a|\tau|} \longleftrightarrow \frac{2a}{a^2+(2\pi f)^2}$$

we have

$$\phi_{gg}(\tau) = \frac{\Omega_p}{2} \cdot \frac{A^2}{\beta} e^{-|\tau|/\beta} \longleftrightarrow \frac{\Omega_p}{2} \frac{A^2}{1+(2\pi f\beta)^2}$$

b) The envelope power, P, is defined as

$$P = E[|g(t)|^2] = E[g_I^2(t)] + E[g_Q^2(t)]$$

But $\phi_{gg}(\tau) = \frac{1}{2} E[g^*(t)\, g(t+\tau)]$

and $\phi_{gg}(0) = \frac{1}{2} E[|g(t)|^2]$

Hence, $P = 2\phi_{gg}(0) = \Omega_p \frac{A^2}{\beta}$

and $\frac{A^2}{\beta} = 1$ or $A = \sqrt{\beta}$.

c) Let $g_1 = g(t)$, $g_2 = g(t+\tau)$

$$\underline{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$ is a complex Gaussian random vector.

In general, a complex Gaussian random vector, $\underline{x}$, completely specified by the mean

$$m_{\underline{x}} = E[\underline{x}]$$

the covariance matrix

$$K_{\underline{x}} = E\left[(\underline{x} - \underline{m}_x)(\underline{x} - \underline{m}_x)^H\right]$$

and the pseudo-covariance matrix

$$J_{\underline{x}} = E\left[(\underline{x} - \underline{m}_x)(\underline{x} - \underline{m}_x)^T\right]$$

where H is conjugate transpose and T is transpose.

For the problem at hand

$$K_{\underline{x}} = E[\underline{g}\,\underline{g}^H]$$

$$= E\begin{bmatrix} g_1 g_1^* & g_1 g_2^* \\ g_2 g_1^* & g_2 g_2^* \end{bmatrix}$$

But $E[g_1 g_1^*] = \Omega_p$

and $E[g_1 g_2^*] = 2\phi_{gg}(\tau) = \Omega_p e^{-|\tau|/\beta} = E[g_2 g_1^*]$

$\therefore$

$$K_{\underline{g}} = \Omega_p \begin{bmatrix} 1 & e^{-|\tau|/\beta} \\ e^{-|\tau|/\beta} & 1 \end{bmatrix}$$

We can also show that $J_{\underline{g}} = [0]$ in a similar manner. For example

$$E[g_1^2] = E\left[(g_I(t) + j g_Q(t))(g_I(t) + j g_Q(t))\right]$$
$$= E[g_I^2(t)] - E[g_Q^2(t)]$$
$$= 0$$

For the case when $J_{\underline{g}} = [0]$, $\underline{g}$ is called proper, and with $m_{\underline{g}} = \underline{0}$, the joint density of $\underline{g}$ has the form

$$f(\underline{g}) = \frac{1}{\pi^n \det(K_{\underline{g}})} e^{-\underline{g}^H K_{\underline{g}}^{-1} \underline{g}}$$

where $\det(\cdot)$ is the matrix determinant.

4.

```
clear all;
fmt = 0.1; % f_m *T
tt = 0:1:399; %t/T
M = 16;
N = (2*M+1)*2;
alpha = 0;
wm = 2*pi*fmt;
beta = pi*(1:M)/M;
i=1;
for t0 = tt,
cos_wn(i,:) = cos(wm*t0*cos(2*pi*(1:M)/N));
xre(i) = sum(2*cos(beta).*cos_wn(i,:));
xim(i) = sum(2*sin(beta).*cos_wn(i,:));
i = i+1;
end;
Xre = xre + sqrt(2)*cos(alpha)*cos(wm*tt);
Xim = xim + sqrt(2)*sin(alpha)*cos(wm*tt);
Xre = Xre/sqrt(2*(M+1));
Xim = Xim/sqrt(2*M);
z = sqrt(Xre.*Xre+Xim.*Xim);
phase = atan2(Xim,Xre)+pi;
zdb = 10*log10(z);
subplot(2,1,1), plot(zdb), title('Envelope (dB)'),
xlabel('time t/T'), ylabel('Envelope (dB)'),
subplot(2,1,2), plot(phase),
title('Phase'),
xlabel('time t/T'), ylabel('Phase'),
axis([0 400 0 2*pi]);
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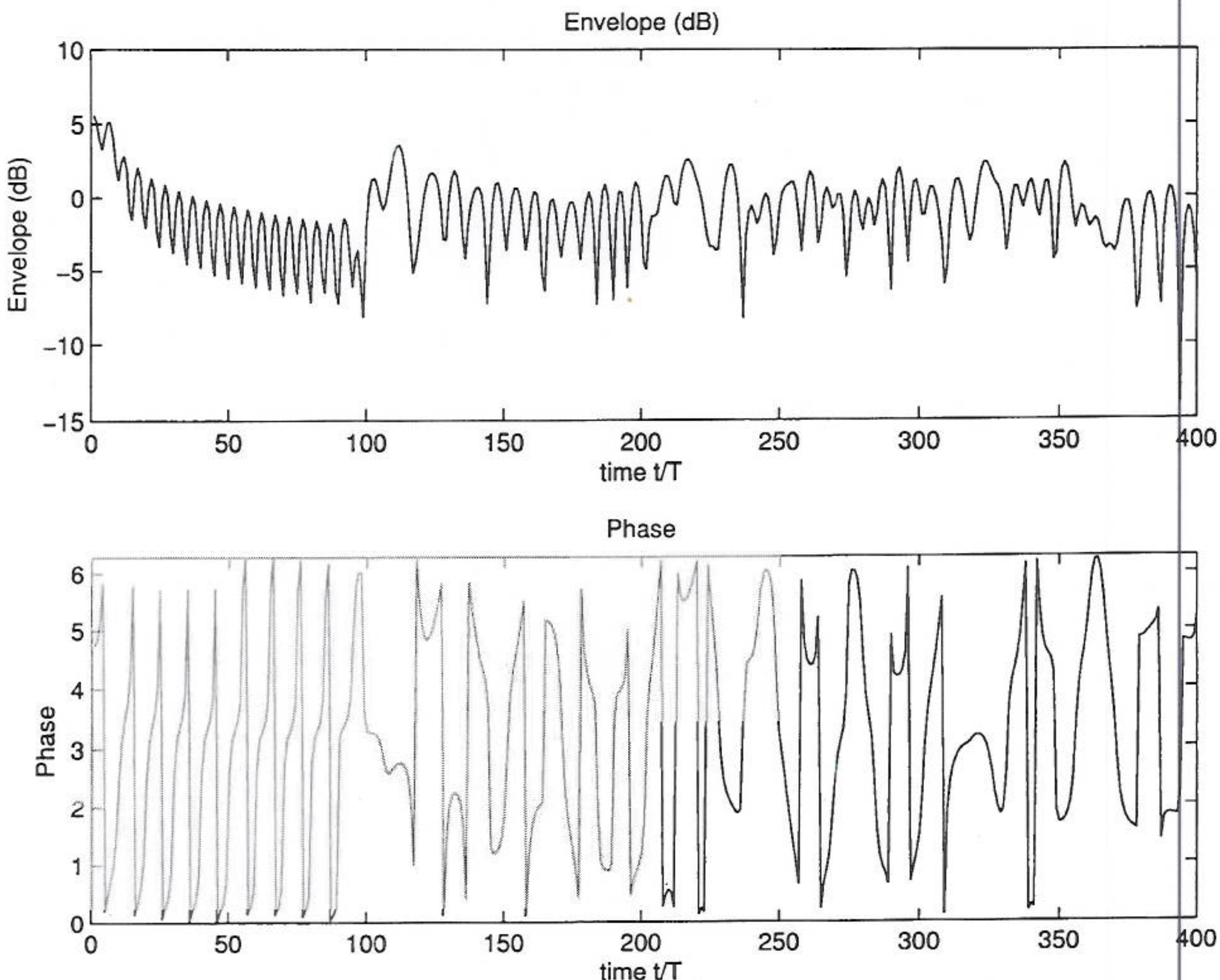
Envelope (dB)
Envelope (dB)
10
5
0
−5
−10
−15
0
50
100
150
200
250
300
350
400
time t/T
Phase
Phase
6
5
4
3
2
1
0
0
50
100
150
200
250
300
350
400
time t/T