ECE 6604 Homework #4 Solutions

1 (a)

$$\Omega_d = -10(3.5)\log_{10}(1000) + \epsilon \text{ dB}$$
$$= -105 \text{ dBm} + \epsilon \text{ dB}$$
$$\Omega_1 = -10(3.5)\log_{10}(3000) + \epsilon_1 \text{ dB}$$
$$= -121.7 \text{ dBm} + \epsilon \text{ dB}$$
$$\Omega_2 = -10(3.5)\log_{10}(4000) + \epsilon_2 \text{ dB}$$
$$= -126.1 \text{ dBm} + \epsilon_2 \text{ dB}$$

$\Omega_d \sim N(-105, 64)$ $\qquad \sigma = 8 \text{ dB}$

$\Omega_1 \sim N(-121.7, 64)$ $\qquad \xi = \frac{\ln 10}{10}$

$\Omega_2 \sim N(-126.1, 64)$

$$\hat{\mu}_1 = \xi \mu_1 (\text{dBm}) = -28.02$$
$$\hat{\mu}_2 = \xi \mu_2 (\text{dBm}) = -29.04$$
$$\hat{\sigma}^2_{\hat{\Omega}} = \xi^2 \sigma_\Omega^2 = 3.39$$

$$\sigma_{\hat{z}}^2 = \ln\left((e^{3.39}-1)\left(\frac{e^{2(-28.02)} + e^{2(-29.03)}}{(e^{-28.02} + e^{-29.03})^2} \right) + 1 \right)$$

$$= 2.91$$

$$\sigma_z^2 = \xi^{-2}\sigma_{\hat{z}}^2 = 54.97$$

$$\mu_{\hat{z}} = \frac{3.39 - 2.91}{2} + \ln\left(e^{-28.02} + e^{-29.03} \right)$$

$$= -27.47$$

$$\mu_z = \xi^{-1}\mu_{\hat{z}} = -119.3 \text{ dBm}$$

Hence $\boxed{\hat{I}_{dBm} \sim N(-119.3,\ 54.97)}$

b) $C_{dBm} \sim N(-105,\ 64)$

$$\left(\frac{C}{\hat{I}}\right)_{dB} = C_{dBm} - I_{dBm} \sim \boxed{N(14.3,\ \underset{\sigma^2}{119})}$$

$\sigma = 10.4$ dB.

3. From the Fenton-Wilkinson method,

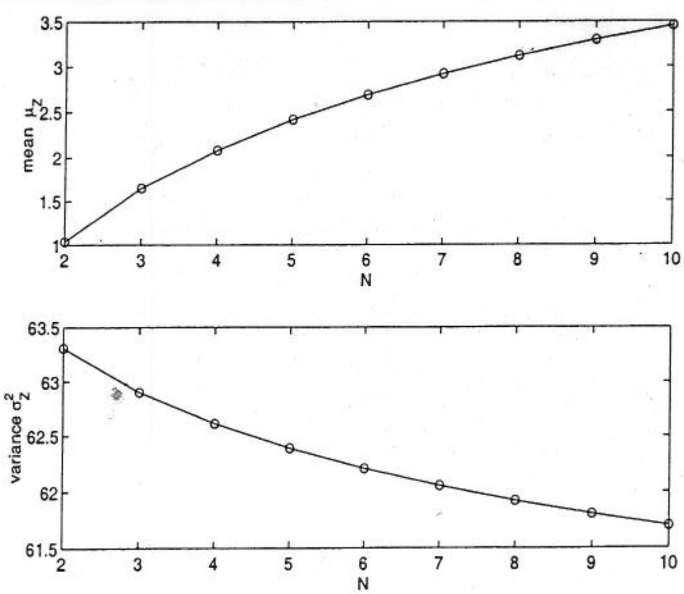
$$\mu_L = \mathrm{E}[L] = \sum_{k=1}^{N} \mathrm{E}[e^{\hat{\Omega}_k}]$$
$$\sigma_L^2 = \mathrm{E}[L^2] - \mu_L^2.$$

Since $e^{\hat{Z}} = \sum_{k=0}^{N} e^{\hat{\Omega}_k} = L$, and $\hat{\Omega}_k$ are independent zero-mean Gaussian random variables with $\sigma_{\hat{\Omega}} = 8\mathrm{dB}$, i.e., $\hat{\Omega}_k \sim \mathrm{N}(0, 64)$, we have,

$$\mu_L = \Big(\sum_{k=1}^{N} e^{\mu_{\hat{\Omega}_k}}\Big) e^{\frac{1}{2}\sigma_{\hat{\Omega}}^2} = e^{\mu_{\hat{Z}} + \frac{1}{2}\sigma_{\hat{Z}}^2}$$
$$\sigma_L^2 = \Big(\sum_{k=1}^{N} e^{2\mu_{\hat{\Omega}_k}}\Big) e^{\sigma_{\hat{\Omega}}^2}(e^{\sigma_{\hat{\Omega}}^2} - 1) = e^{2\mu_{\hat{Z}}} e^{\sigma_{\hat{Z}}^2}(e^{\sigma_{\hat{Z}}^2} - 1)$$

Then,

$$\sigma_{\hat{Z}}^2 = \ln\Big((e^{\sigma_{\hat{\Omega}}^2} - 1)\frac{\sum_{k=1}^{N} e^{2\mu_{\hat{\Omega}_k}}}{(\sum_{k=1}^{N} e^{\mu_{\hat{\Omega}_k}})^2} + 1\Big) = \ln(1 + (e^{64} - 1)\frac{N}{N^2}) = \ln(1 + (e^{64} - 1)/N)$$
$$\mu_{\hat{Z}} = \frac{\sigma_{\hat{\Omega}}^2 - \sigma_{\hat{Z}}^2}{2} + \ln(\sum_{k=1}^{N} e^{\mu_{\hat{\Omega}_k}}) = \frac{64 - \sigma_{\hat{Z}}^2}{2} + \ln N$$



Mean $\mu_{\hat{Z}}$ and variance $\sigma_{\hat{Z}}^2$ as a function of N.

3. Fading model:

Desired signal (Rician), from (2.51)

$$P_x(x) = \frac{K+1}{\Omega_0} \exp\left\{-K - \frac{(K+1)}{\Omega_0} x\right\} I_0\left(2\sqrt{\frac{K(K+1)}{\Omega_0} x}\right) = \frac{1}{\Omega_0/(K+1)} \exp\left\{-\frac{2x + \frac{2K\Omega_0}{K+1}}{2\left(\frac{\Omega_0}{K+1}\right)}\right\} I_0\left(\frac{\sqrt{2x} \cdot \sqrt{\frac{2K\Omega_0}{K+1}}}{\frac{\Omega_0}{K+1}}\right) \quad , x \geqslant \emptyset$$

where $\sigma_x^2 \triangleq \frac{\Omega_0}{K+1}$ is the diffused signal component
$\frac{S^2}{2} \triangleq \frac{K\Omega_0}{K+1}$ is the direct LOS signal component
} mean signal power is Ω_0

Cochannel single interferer (Rayleigh), from (2.44)

$$P_{y_1}(y_1) = \frac{1}{\Omega_1} \exp\left(-\frac{y_1}{\Omega_1}\right) \quad , y_1 \geqslant 0 \qquad \text{with mean power } \Omega_1$$

$$O = 1 - P_r\left(\frac{x}{y_1} > \lambda_{th}\right) = 1 - \int_0^\infty P_x(x)dx \int_0^{\frac{x}{\lambda_{th}}} P_{y_1}(y_1)dy_1 = 1 - \int_0^\infty P_x(x)dx \cdot \left[1 - e^{-\frac{x}{\lambda_{th}\Omega_1}}\right] = \int_0^\infty e^{-\frac{x}{\lambda_{th}\Omega_1}} P_x(x)dx$$

$$\Longrightarrow O = E\left[e^{-\frac{x}{\lambda_{th}\Omega_1}}\right] \quad ; \quad \text{but } E[e^{rx}] = \frac{1}{1 - r\sigma_x^2} \exp\left\{\frac{r\frac{S^2}{2}}{1 - r\sigma_x^2}\right\}$$

$$\Longrightarrow O = \frac{1}{1 + \frac{1}{\lambda_{th}\Omega_1}\left(\frac{\Omega_0}{K+1}\right)} \exp\left\{\frac{\left(-\frac{1}{\lambda_{th}\Omega_1}\right)\frac{K\Omega_0}{K+1}}{1 + \frac{1}{\lambda_{th}\Omega_1}\left(\frac{\Omega_0}{K+1}\right)}\right\} = \frac{\lambda_{th}}{\lambda_{th} + b_1} \exp\left\{-\frac{Kb_1}{\lambda_{th} + b_1}\right\} \qquad \text{with } b_1 = \frac{\Omega_0}{(K+1)\Omega_1}$$

4a) The inner product is

$$(s_1, s_2) = A^2 \int_0^T \cos 2\pi f_c t \cos 2\pi (f_c + \Delta f) t \, dt$$

$$= \frac{A^2}{2} \int_0^T \cos(4\pi f_c t + 2\pi \Delta f t) dt + \frac{A^2}{2} \int_0^T \cos 2\pi \Delta f t \, dt$$

$$= \frac{A^2}{2} \left. \frac{\sin(4\pi f_c + 2\pi \Delta f)t}{4\pi f_c + 2\pi \Delta f} \right|_0^T + \frac{A^2}{2} \left. \frac{\sin 2\pi \Delta f t}{2\pi \Delta f} \right|_0^T$$

$$= \frac{A^2 T}{2} \frac{\sin(4\pi f_c + 2\pi \Delta f)T}{4\pi f_c T + 2\pi \Delta f T} + \frac{A^2 T}{2} \frac{\sin 2\pi \Delta f T}{2\pi \Delta f T}$$

Since $f_c T \gg 1$, the first term is approximately zero.

$$(s_1, s_2) = \frac{A^2 T}{2} \frac{\sin 2\pi \Delta f T}{2\pi \Delta f T} = 0 \implies \Delta f = \frac{k}{2T}, \quad k \text{ a non-zero integer.}$$

Hence, the smallest Δf is $\Delta f = 1/2T$

b) $$(s_1, s_2) = A^2 \int_0^T \cos(2\pi f_c t + \phi_1)\cos(2\pi(f_c + \Delta f)t + \phi_2)\,dt$$

$$= \frac{A^2}{2}\int_0^T \cos(4\pi f_c t + 2\pi\Delta f t + \phi_1 + \phi_2)\,dt + \frac{A^2}{2}\int_0^T \cos(2\pi\Delta f t + \phi_2 - \phi_1)\,dt$$

$$= \frac{A^2}{2}\,\frac{\sin(4\pi f_c t + 2\pi\Delta f t + \phi_1 + \phi_2)}{4\pi f_c + 2\pi\Delta f}\Bigg|_0^T + \frac{A^2}{2}\,\frac{\sin(2\pi\Delta f t + \phi_2 - \phi_1)}{2\pi\Delta f}\Bigg|_0^T$$

$$= \frac{A^2 T}{2}\left\{\frac{\sin(4\pi f_c T + 2\pi\Delta f T + \phi_1 + \phi_2) - \sin(\phi_1 + \phi_2)}{4\pi f_c T + 2\pi\Delta f T} + \frac{\sin(2\pi\Delta f T + \phi_2 - \phi_1) - \sin(\phi_2 - \phi_1)}{2\pi\Delta f T}\right.$$

Since $f_c T \gg 1$, the first term is approximately zero

$$(s_1, s_2) = \frac{A^2 T}{2}\,\frac{\sin(2\pi\Delta f T + \phi_2 - \phi_1) - \sin(\phi_2 - \phi_1)}{2\pi\Delta f T}$$

$$= \frac{A^2 T}{2}\,\frac{\sin(2\pi\Delta f T)\cos(\phi_2 - \phi_1) + \cos(2\pi\Delta f T)\sin(\phi_2 - \phi_1) - \sin(\phi_2 - \phi_1)}{2\pi\Delta f T}$$

$$= 0 \implies \Delta f = \frac{k}{T}, \quad k \text{ a non-zero integer}$$

Hence, the smallest Δf is $\Delta f = 1/T$.