ECE 6604 Homework #2 Solutions

1 a)

$$\phi_{rr}(\tau) = E[r(t)r(t+\tau)]$$
$$= E\Big[(g_I(t)\cos 2\pi f_c t - g_Q(t)\sin 2\pi f_c t) \times (g_I(t+\tau)\cos 2\pi f_c(t+\tau) - g_Q(t+\tau)\sin 2\pi f_c(t+\tau))\Big]$$

$$= E[g_I(t)g_I(t+\tau)]\cos 2\pi f_c t \cos 2\pi f_c(t+\tau)$$
$$- E[g_Q(t)g_I(t+\tau)]\sin 2\pi f_c t \cos 2\pi f_c(t+\tau)$$
$$- E[g_I(t)g_Q(t+\tau)]\cos 2\pi f_c t \sin 2\pi f_c(t+\tau)$$
$$+ E[g_Q(t)g_Q(t+\tau)]\sin 2\pi f_c t \sin 2\pi f_c(t+\tau)$$

use trig identities

$$\cos A\cos B = \tfrac{1}{2}\cos(A-B) + \tfrac{1}{2}\cos(A+B)$$
$$\sin A\sin B = \tfrac{1}{2}\cos(A-B) - \tfrac{1}{2}\cos(A+B)$$
$$\sin A\cos B = \tfrac{1}{2}\sin(A-B) + \tfrac{1}{2}\sin(A+B)$$

Then

$$\phi_{rr}(\tau) = \tfrac{1}{2}\phi_{g_Ig_I}(\tau)\left[\cos(2\pi f_c\tau) + \cos(2\pi f_c(2t+\tau))\right]$$
$$-\tfrac{1}{2}\phi_{g_Qg_I}(\tau)\left[-\sin(2\pi f_c\tau) + \sin(2\pi f_c(2t+\tau))\right]$$
$$-\tfrac{1}{2}\phi_{g_Ig_Q}(\tau)\left[\sin(2\pi f_c\tau) + \sin(2\pi f_c(2t+\tau))\right]$$
$$+\tfrac{1}{2}\phi_{g_Qg_Q}(\tau)\left[\cos(2\pi f_c\tau) - \cos(2\pi f_c(2t+\tau))\right]$$

Rearranging gives.

$$\phi_{rr}(\tau) = \frac{1}{2}\left[\phi_{g_I g_I}(\tau) + \phi_{g_Q g_Q}(\tau)\right]\cos 2\pi f_c \tau$$
$$+ \frac{1}{2}\left[\phi_{g_I g_I}(\tau) - \phi_{g_Q g_Q}(\tau)\right]\cos(2\pi f_c(t+\tau))$$
$$- \frac{1}{2}\left[\phi_{g_I g_Q}(\tau) - \phi_{g_Q g_I}(\tau)\right]\sin 2\pi f_c \tau$$
$$- \frac{1}{2}\left[\phi_{g_I g_Q}(\tau) + \phi_{g_Q g_I}(\tau)\right]\sin(2\pi f_c(t+\tau)) \quad (1)$$

Since $r(t)$ is assumed WSS, we must have $\phi_{rr}(\tau)$ a function of τ only. Hence

$$\phi_{g_I g_I}(\tau) = \phi_{g_Q g_Q}(\tau) \quad (2)$$
$$\phi_{g_I g_Q}(\tau) = -\phi_{g_Q g_I}(\tau)$$

b) With the result of part a), we have

$$\phi_{rr}(\tau) = \phi_{g_I g_I}(\tau)\cos 2\pi f_c \tau - \phi_{g_I g_Q}(\tau)\sin 2\pi f_c \tau$$

by substituting (2) into (1).

2. $g(t,\tau) = U_T(\tau)\cos(\Omega t + \phi_0)\,, \quad 0 \le \tau \le T$

$T = 10\,\mu s\,, \quad \Omega = 10\pi$

a) $T(f,t) = \mathcal{F}\{g(t,\tau)\}$

$= \cos(\Omega t + \phi_0)\,\mathcal{F}\{U_T(t)\}$

$= T\,\mathrm{sinc}(\pi f T)\,e^{j\pi f T}\cos(\Omega t + \phi_0)$

b) $\tilde{r}(t) = g(t,\tau) * \tilde{s}(t)$

$= \int_0^t g(t,\tau)\,\tilde{s}(t-\tau)\,d\tau$

$= \int_0^t U_T(\tau)\cos(\Omega t + \phi_0)\,\tilde{s}(t-\tau)\,d\tau$

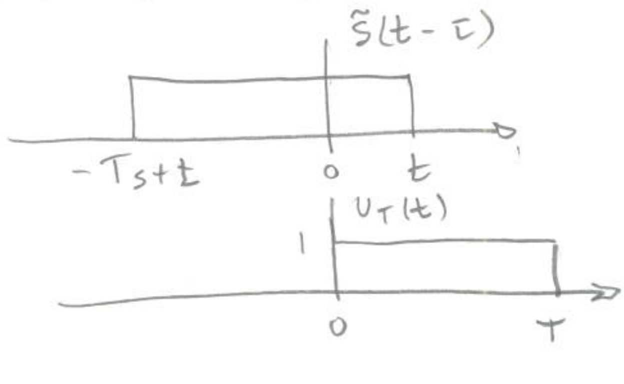
where

$$U_T(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{else} \end{cases}$$

$$\tilde{s}(t) = \begin{cases} 1, & 0 \le t \le T_s \\ 0, & \text{else} \end{cases}$$

The problem can be solved using graphical convolution. There are 2 cases to consider i) $T < T_s$ and ii) $T > T_s$

Case i $T < T_s$



i) For $t \le 0$, $\tilde{r}(t) = 0$

ii) For $0 \le t \le T$, $\tilde{r}(t) = \{\cos(\Omega t + \phi_0)\} \int_0^t d\tau$

$$= t\cos(\Omega t + \phi_0)$$

iii) For $T \le t \le T_s$, $\tilde{r}(t) = T\cos(\Omega t + \phi_0)$

iv) For $T_s \le t \le T + T_s$, $\tilde{r}(t) = (T + T_s - t)\cos(\Omega t + \phi_0)$

v) For $t > T + T_s$, $\tilde{r}(t) = 0$

Summary

$$\tilde{r}(t) = \begin{cases} 0, & t \le 0 \\ t\cos(\Omega t + \phi_0), & 0 \le t \le T \\ T\cos(\Omega t + \phi_0), & T \le t \le T_s \\ (T + T_s - t)\cos(\Omega t + \phi_0), & T_s \le t \le T + T_s \\ 0, & t > T + T_s \end{cases}$$

Case ii $T > T_s$: Similar approach, with the final result.

$$\tilde{r}(t) = \begin{cases} 0, & t < 0 \\ t\cos(\Omega t + \phi_0), & 0 \le t \le T_s \\ T_s\cos(\Omega t + \phi_0), & T_s \le t \le T \\ (T + T_s - t)\cos(\Omega t + \phi_0), & T \le t \le T + T_s \\ 0, & t > T + T_s \end{cases}$$

3. Given downlink band

$$935 \le f_c \le 960 \text{ MHz}$$

Uplink band

$$890 \le f_c \le 915 \text{ MHz}$$

and train speed 250 Km/h

we have

$$f_m = \frac{v}{\lambda_c} = \frac{v}{c} f_c$$

Uplink:

$$f_m = \frac{250 \times 10^3}{60 \cdot 60} \cdot \frac{1}{3 \times 10^8} \cdot 960 \times 10^6$$

$$= 222.2 \text{ Hz.}$$

Downlink

$$f_m = \frac{250 \times 10^3}{60 \cdot 60} \cdot \frac{1}{3 \times 10^8} \cdot 915 \times 10^6$$

$$= 211.8 \text{ Hz}$$

4. Using an approach similar to Eq (2.30) in the text, we can write

$$S_{gg}(f)\,|df| = \frac{\Omega_p}{2}\left\{G(\theta)p(\theta) + G(-\theta)p(-\theta)\right\}|d\theta|$$

The Doppler frequency associated with an incident plane wave at angle θ is

$$f = f_m \cos\theta$$

and, hence,

$$|df| = f_m\,|-\sin\theta\, d\theta| = \sqrt{f_m^2 - f^2}\;|d\theta|$$

$$\therefore\quad S_{gg}(f) = \frac{\Omega_p/2}{\sqrt{f_m^2 - f^2}}\left\{G(\theta)p(\theta) + G(-\theta)p(-\theta)\right\}$$

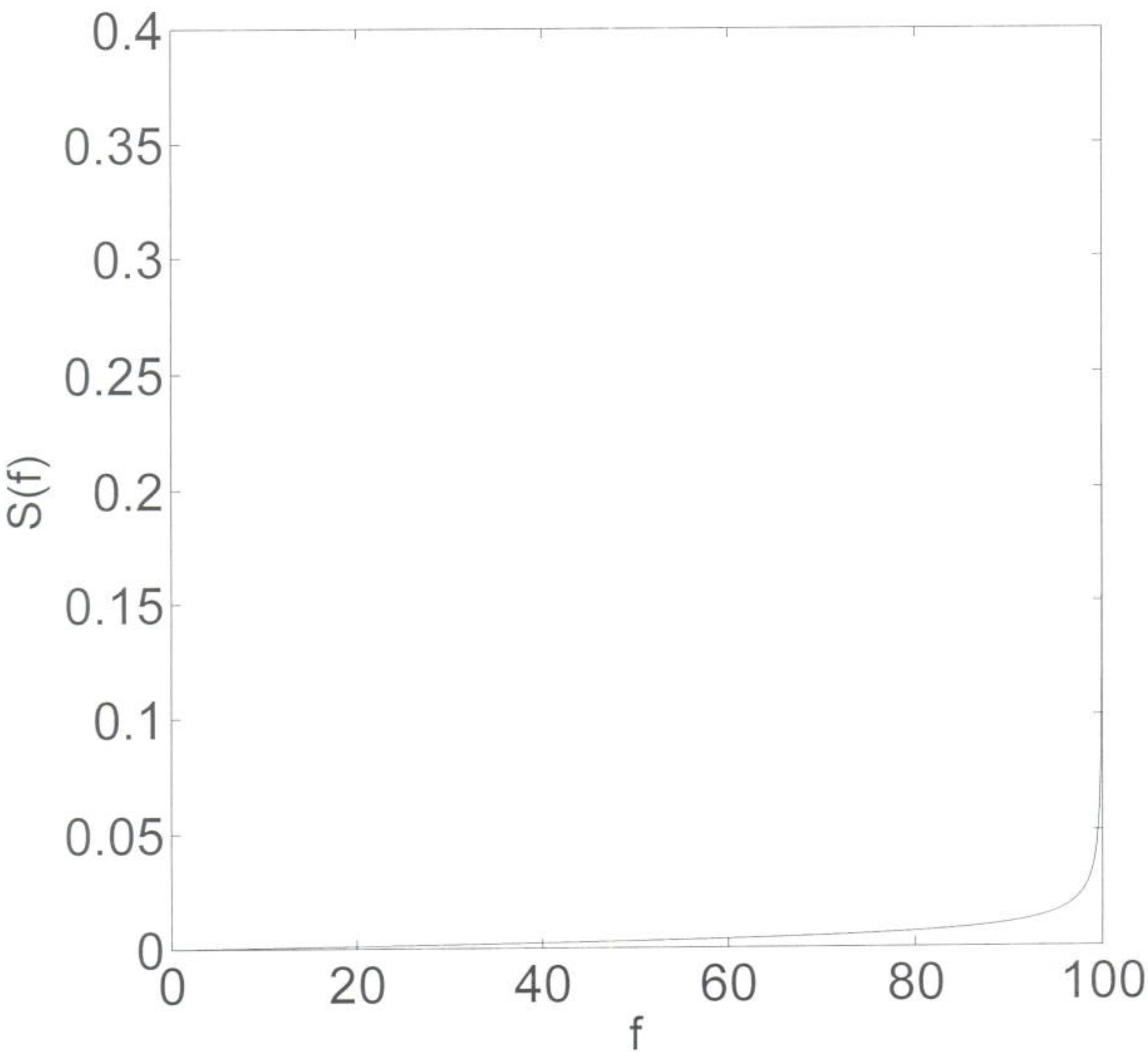
$$\theta = \cos^{-1}(f/f_m)$$

For $G(\theta)=1$ and $p(\theta)$ as given

$$S_{gg}(f) = \frac{\Omega_p/2}{\sqrt{f_m^2 - f^2}} \cdot \frac{\pi}{4|\theta_m|}\cos\left(\frac{\pi}{2}\cdot\frac{\theta}{\theta_m}\right)$$

where $0 \le |\theta| \le |\theta_m|$ and $\theta = \cos^{-1}(f/f_m)$

Example for $\theta_m = \pi/2$, $f_m = 100$
Note that $S_{gg}(f)$ has only
positive frequencies in this case.



5/ For $f_c = 5.7$ GHz and $v = 80$ Km/h

$$f_m = \frac{v}{c} \cdot f_c = \frac{80\times10^3/(60\times60)}{3\times10^8}(5.7\times10^9)$$

$$= 422.2 \text{ Hz}$$

1. $L_R = \sqrt{2\pi}\, f_m \rho e^{-\rho^2}$ where $\rho = \frac{R}{R_{rms}} = 1$

$$= \sqrt{2\pi} \times 422.2 \cdot 1 \cdot e^{-1}$$

$$= 389.3 \text{ crossing/sec}$$

So in 5 seconds $389.3 \times 5 = 1946.7$ crossings

2. $$\bar{t} = \frac{e^{\rho^2} - 1}{\rho f_m \sqrt{2\pi}} = \frac{e - 1}{422.2\sqrt{2\pi}}$$

$$= 1.62 \text{ ms}$$

3. For $\rho = -20$ dB $\quad \rho = 10^{\rho_{dB}/20} = 0.1$

$$\bar{t} = \frac{e^{\rho^2} - 1}{\rho f_m \sqrt{2\pi}} = \frac{e^{(0.1)^2} - 1}{\rho\, 422.2\sqrt{2\pi}}$$

$$= 94.9\,\mu s.$$