ECE 6604 Homework #3 Solutions. 1/

1/ 9. Since $K=0$, $s=0$

$$p(\alpha,\dot\alpha) = \frac{\alpha\,(2\pi)^{-1/2}}{\sqrt{Bb_0}}\, e^{-\frac{1}{2Bb_0}\left[B\alpha^2 + b_0^2\dot\alpha^2\right]}$$

$$L_R = \int_0^\infty \dot\alpha\, p(R,\dot\alpha)\, d\dot\alpha$$

$$= \frac{R\,(2\pi)^{-1/2}}{\sqrt{Bb_0}}\, e^{-R^2/2b_0}\int_0^\infty \dot\alpha\, e^{-\frac{b_0\dot\alpha^2}{2B}}\, d\dot\alpha$$

$$= \frac{\sqrt{B}\cdot R e^{-R^2/2b_0}}{\sqrt{2\pi b_0^3}}$$

$$= \sqrt{\frac{b_2 b_0 - b_1^2}{b_0^2}}\cdot\frac{R}{\sqrt{2b_0}\cdot\sqrt{\pi}}\, e^{-R^2/2b_0}$$

$$= \sqrt{\frac{b_2}{b_0} - \frac{b_1^2}{b_0^2}}\cdot\frac{\rho}{\sqrt{\pi}}\, e^{-\rho^2} \quad ; \quad \rho = \frac{R}{\sqrt{2b_0}} = \frac{R}{\sqrt{\Omega_p}}$$

2/ 13. $\phi_T(f,m;t,s) = \phi_T(f,m;s+\Delta t,s) = \phi_T(f,m;\Delta t)$

$$\phi_H(f,m;\nu,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_T(f,m;t,s)\, e^{-j2\pi(\nu t-\mu s)}\,dt\,ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_T(f,m;\Delta t)\, e^{-j2\pi(\nu s-\mu s+\nu\Delta t)}\,ds\,d\Delta t$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi(\nu-\mu)s}\,ds\int_{-\infty}^{\infty}\phi_T(f,m;\Delta t)\, e^{-j2\pi\nu\Delta t}\,d\Delta t$$

$$= \delta(\nu-\mu)\,\Psi_H(f,m;\nu)$$

$\phi_g(t,s;\tau,\eta) = \phi_g(s+\Delta t,s;\tau,\eta) = \phi_g(\Delta t;\tau,\eta)$

$$\phi_S(\tau,\eta;\nu,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_g(t,s;\tau,\eta)\, e^{-j2\pi(\nu t-\mu s)}\,dt\,ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_g(\Delta t;\tau,\eta)\, e^{-j2\pi(\nu s-\mu s+\nu\Delta t)}\,ds\,d\Delta t$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi(\nu-\mu)s}\,ds\int_{-\infty}^{\infty}\phi_g(\Delta t;\tau,\eta)\, e^{-j2\pi\nu\Delta t}\,d\Delta t$$

$$= \delta(\nu-\mu)\,\Psi_S(\tau,\eta;\nu)$$

Singularity with respect to the Doppler variable implies that multipath components arriving with different Doppler shift and, hence, direction have uncorrelated fading.

3/ $\phi_g(\tau) = e^{-\tau/T} \quad \tau \geq 0$

$$\mu_\tau = \frac{\int_0^\infty \tau \phi_g(\tau)\, d\tau}{\int_0^\infty \phi_g(\tau)\, d\tau}$$

$$\int_0^\infty \tau \phi_g(\tau)\, d\tau = \int_0^\infty \tau e^{-\tau/T} d\tau = \frac{1}{\left(\frac{1}{T}\right)^2} = T^2$$

$$\int_0^\infty \phi_g(\tau)\, d\tau = \int_0^\infty e^{-\tau/T} d\tau = \frac{1}{(1/T)} = T$$

$$\mu_\tau = T = 10\mu s$$

$$\sigma_\tau^2 = \frac{\int_0^\infty (\tau - \mu_\tau)^2 \phi_g(\tau)\, d\tau}{\int_0^\infty \phi_g(\tau)\, d\tau}$$

$$\int_0^\infty (\tau - \mu_\tau)^2 \phi_g(\tau)\, d\tau = \int_0^\infty \tau^2 \phi_g(\tau)\, d\tau - 2\mu_\tau \int_0^\infty \tau \phi_g(\tau)\, d\tau + \mu_\tau^2 \int_0^\infty \phi_g(\tau)\, d\tau$$

$$= \frac{2}{(1/T)^3} - 2T \cdot T^2 + T^3$$

$$= 2T^3 - 2T^3 + T^3$$

$$= T^3$$

$$\sigma_\tau^2 = \frac{T^3}{T} = T^2$$

$$\sigma_\tau = T = 10\mu s$$

$$B_c \propto \frac{1}{T} = 100 \text{ KHz}$$

4. a) Delay psd is $\psi_g(\tau)$

$$\psi_g(\tau) = \int_{-\infty}^{\infty} \psi_s(\tau,\nu)\,d\nu$$

$$= \psi_1(\tau)\int_{-\infty}^{\infty} \psi_2(\nu)\,d\nu$$

But $\int_{-\infty}^{\infty} \psi_2(\nu)\,d\nu = 2\int_0^{\nu_m} \frac{1}{\nu_m}\left[1-\frac{\nu^2}{\nu_m^2}\right]d\nu$

$$= 2\frac{\nu}{\nu_m} - \frac{2\nu^3}{3\nu_m^3}\Big|_0^{\nu_m}$$

$$= 2 - 2/3 = 4/3$$

$$\psi_g(\tau) = \begin{cases} \frac{4}{3}, & 0 \le \tau \le 100\text{ ms} \\ 0, & \text{else} \end{cases}$$

b) Doppler psd

$$\phi_H(\nu) = \int_{-\infty}^{\infty} \psi_s(\tau,\nu)\,d\tau$$

$$= \psi_2(\nu)\int_{-\infty}^{\infty} \psi_1(\tau)\,d\tau$$

$$= 0.1\,\psi_2(\nu)$$

$$= \begin{cases} \frac{0.1}{\nu_m}\left[1-(\nu/\nu_m)^2\right], & 0 \le |\nu| \le \nu_m \\ 0, & \text{else} \end{cases}$$

c)

$$\mu_\tau = \frac{3}{0.1}\int_0^{0.1} \frac{\tau}{3}\,d\tau = \frac{3}{0.1}\,\frac{\tau^2}{2(3)}\Big|_0^{0.1}$$

$$= \frac{(0.1)^2}{(0.1)(2)} = 50 \text{ ms}$$

$$\int_0^{0.1} \frac{\tau^2}{3}\,d\tau = \frac{\tau^3}{9}\Big|_0^{0.1} = \frac{(0.1)^3}{9}$$

$$\sigma_\tau^2 = \frac{(0.1)^3}{9} \times \frac{3}{0.1} - (0.05)^2$$

$$= \frac{0.01}{3} - (0.05)^2$$

$$= 833 \times 10^{-6}$$

$$\sigma_\tau = 28.87 \text{ ms}$$

d) $\nu_m = 10$ Hz

$$\mu_\nu = \frac{3}{0.1}\int_{-\nu_m}^{\nu_m} \nu\,\phi_H(\nu)\,d\nu = 0$$

$$\sigma_\nu^2 = (3)(2)\int_0^{\nu_m} \frac{\nu^2}{\nu_m}\left[1 - \frac{\nu^2}{\nu_m^2}\right]d\nu$$

$$= 6\left[\frac{\nu^3}{3\nu_m} - \frac{\nu^5}{5\nu_m^3}\right]_0^{\nu_m}$$

$$= 6\left[\frac{\nu_m^2}{3} - \frac{\nu_m^2}{5}\right]$$

$$= 0.8\,\nu_m^2$$

$$\therefore \sigma_\nu = 0.894\,\nu_m = 8.94 \text{ Hz}$$

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a) $$\psi_g(\Delta t : \tau) \underset{F^{-1}}{\overset{F}{\rightleftarrows}} \phi_T(\Delta f : \Delta t)$$

Transform is w.r.t. τ or Δf

$$\frac{1}{a + j2\pi\Delta f} \longleftrightarrow e^{-a\tau}u(\tau) \quad ; \; u(\tau) = \begin{cases} 1, & \tau \geq 0 \\ 0, & \text{else} \end{cases}$$

Hence,

$$\psi_g(\Delta t : \tau) = e^{-b|\Delta t|}\, e^{-a\tau}\, u(\tau)$$

b) $$\psi_g(\Delta t : \tau) \underset{F^{-1}}{\overset{F}{\rightleftarrows}} \psi_s(\nu ; \tau)$$

Transform is w.r.t Δt or ν

$$e^{-b|\Delta t|} \longleftrightarrow \frac{2b}{b^2 + (2\pi\nu)^2}$$

Hence,

$$\psi_s(\nu ; \tau) = \frac{2b}{b^2 + (2\pi\nu)^2} \cdot e^{-a\tau}u(\tau)$$

c) $$\psi_g(\tau) = e^{-b}e^{-a\tau}u(\tau)$$

$$\int_0^\infty \psi_g(\tau)\,d\tau = \frac{e^{-b}}{a} \qquad \int_0^\infty \tau\,\psi_g(\tau)\,d\tau = \frac{e^{-b}}{a^2}$$

$$\mu_\tau = \frac{\int_0^\infty \tau\,\psi_g(\tau)\,d\tau}{\int_0^\infty \psi_g(\tau)\,d\tau} = \frac{1}{a}$$