ECE 6604 Assignment #7 Solutions

6.7 a) DPSK $\quad P_b(\gamma_b) = \frac{1}{2}e^{-\gamma_b} \quad ; \quad \gamma_b = \alpha^2 \frac{E_b}{N_o}$

$$\bar{P}_b = \int_0^\infty P_b(x)\, p_{\gamma_b}(x)\, dx$$

But $\quad p_{\gamma_b}(x) = 0.2\,\delta(x) + 0.5\,\delta(x - E_b/N_o) + 0.3\,\delta(x - 4E_b/N_o)$

$$\bar{P}_b = 0.1 + 0.25 e^{-E_b/N_o} + 0.15 e^{-4E_b/N_o}$$

Let $\quad \bar{\gamma}_b = E[\gamma_b] = (0.2)(0) + \left(\frac{E_b}{N_o}\right)(0.5) + \left(\frac{4E_b}{N_o}\right)(0.3) = 1.7\frac{E_b}{N_o}$

$$\bar{P}_b = 0.1 + 0.25 e^{-\bar{\gamma}_b/1.7} + 0.15 e^{-\frac{4\bar{\gamma}_b}{1.7}}$$

As $\bar{\gamma}_b \to \infty$, $\bar{P}_b \to 0.1$

b) $\quad \gamma_b = \max(\gamma_1, \gamma_2) \quad ; \quad \gamma_1 = \alpha_1^2 \frac{E_b}{N_o} \quad , \quad \gamma_2 = \alpha_2^2 \frac{E_b}{N_o}$

$F_{\gamma_b}(x) = F_{\gamma_1}(x) \cdot F_{\gamma_2}(x) = \left[F_{\gamma_1}(x)\right]^2$ since α_1 and α_2 are iid.

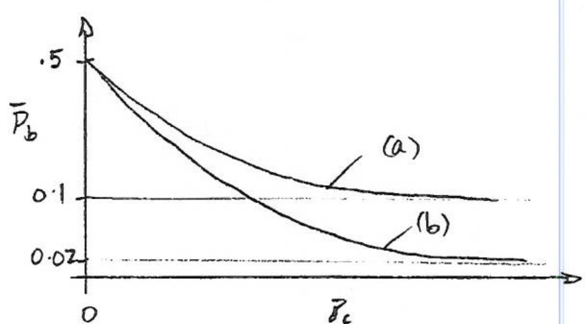
$p_{\gamma_b}(x) = \frac{d}{dx} F_{\gamma_b}(x) = 0.04\,\delta(x) + 0.45\,\delta(x - E_b/N_o) + 0.51\,\delta(x - 4E_b/N_o)$

$$\bar{P}_b = 0.02 + 0.225 e^{-E_b/N_o} + 0.255 e^{-4E_b/N_o}$$

$\bar{\gamma}_c = 1.7\frac{E_b}{N_o}$ as before

$$\bar{P}_b = 0.02 + 0.225 e^{-\bar{\gamma}_c/1.7} + 0.255 e^{-4\bar{\gamma}_c/1.7}$$

As $\bar{\gamma}_c \to \infty$, $\bar{P}_b \to 0.02$

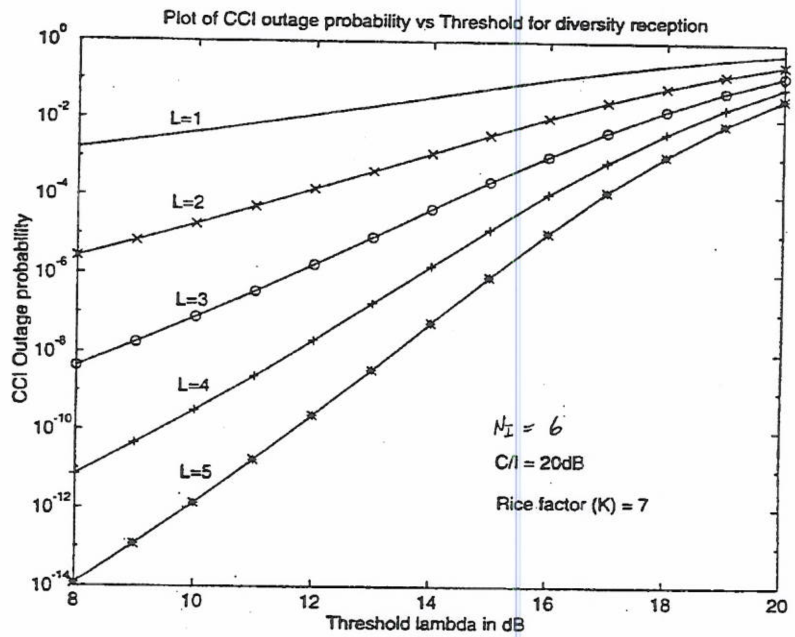


For case (a) $\bar{\gamma}_b = \bar{\gamma}_c$

2/ 6.14 $O = P_r(\lambda_S < \lambda_{th}) = P_r(\lambda_1 < \lambda_{th}, \lambda_2 < \lambda_{th}, \cdots, \lambda_L < \lambda_{th}) = \left[P_r(\lambda_1 < \lambda_{th})\right]^L$, ⟨since λ_i's are i.i.d.⟩

where $P_r(\lambda_1 < \lambda_{th}) = \frac{\lambda_{th}}{\lambda_{th}+b_1} e^{-\frac{Kb_1}{\lambda_{th}+b_1}} \left\{ \sum_{k=0}^{N_I-1} \left(\frac{b_1}{\lambda_{th}+b_1}\right)^k \sum_{m=0}^{k} \binom{k}{m} \frac{1}{m!} \left(\frac{K\lambda_{th}}{\lambda_{th}+b_1}\right)^m \right\}$

$b_1 = \frac{\Omega_0}{(K+1)\Omega_1} = \frac{N_I \Lambda}{(K+1)}$, $\Lambda \triangleq CIR = \frac{\Omega_0}{N_I \Omega_1}$ ⟨see Problem * 3.7 for derivation⟩



3/ 6.15 a) We will find the cdf of λ as follows:

$$
\begin{aligned}
F_\lambda(x) &= P(\lambda = \frac{s_0}{s_1} \le x) = \int_0^\infty \int_0^{xs_1} p_{s_0}(s_0) p_{s_1}(s_1) ds_0 ds_1 \\
&= \int_0^\infty \int_0^{xs_1} \frac{1}{\bar{s}_0} e^{-s_0/\bar{s}_0} \frac{1}{\bar{s}_1} e^{-s_1/\bar{s}_1} ds_0 ds_1 \\
&= \frac{1}{\bar{s}_1} \int_0^\infty e^{-s_1/\bar{s}_1} \left\{ -e^{-s_0/\bar{s}_0} \Big|_0^{xs_1} \right\} ds_1 = \frac{1}{\bar{s}_1} \int_0^\infty e^{-s_1/\bar{s}_1} \left[1 - -e^{-xs_1/\bar{s}_0} \right] ds_1 \\
&= \frac{1}{\bar{s}_1} \int_0^\infty e^{-s_1/\bar{s}_1 ds_1} - \frac{1}{\bar{s}_1} \int_0^\infty e^{-s_1 \left(\frac{1}{\bar{s}_1} + \frac{x}{\bar{s}_1} \right)} ds_1 \\
&= 1 - \frac{1}{1 + \frac{\bar{s}_1}{\bar{s}_0} x} = \frac{x}{\frac{\bar{s}_0}{\bar{s}_1} + x} = \frac{x}{\bar{\lambda} + x}
\end{aligned}
$$

The pdf of λ is just the derivative of its cdf and is

$$p_\lambda(x) = \frac{\bar{\lambda}}{(x + \bar{\lambda})^2} \text{ for } \lambda \ge 0.$$

b) The mean value of λ is

$$E[\lambda] = \int_0^\infty x p_\lambda(x) dx = \int_0^\infty \frac{x\bar{\lambda}}{(x + \bar{\lambda})^2} dx = \infty.$$

c) When the selection diversity combining is used the output of the selection combiner will be

$$\lambda_b^s = \max\{\lambda_1, \ldots, \lambda_L\}$$

The cdf of the signal-to-interference ratio at the output of the selection combiner will be

$$F_{\lambda_b^s}(x) = \left(\frac{x}{x+\overline{\lambda}}\right)^L.$$

The pdf of λ_b^s will be the derivative of its cdf, i.e.

$$p_{\lambda_b^s}(x) = L\left(\frac{x}{x+\overline{\lambda}}\right)^{L-1} \frac{\overline{\lambda}}{(x+\overline{\lambda})^2}$$

7/6.17

$$J = 2\underline{w}^T \Phi_{\tilde{r}_t\tilde{r}_t} \underline{w}^* - 2\,\mathrm{Re}\{\Phi_{\tilde{s}_0\tilde{r}_t}\underline{w}^*\} - 2E_{av}$$

For the gradient of real-valued J w.r.t complex $\underline{w}$

$$\nabla_{\underline{w}} J = \nabla_{\underline{w}_R} J + j\nabla_{\underline{w}_I} J$$

$$\Phi_{\tilde{r}_t\tilde{r}_t} = \Phi_{\tilde{r}_t\tilde{r}_t R} + j\Phi_{\tilde{r}_t\tilde{r}_t I}$$

$$\Phi_{\tilde{s}_0\tilde{r}_t} = \Phi_{\tilde{s}_0\tilde{r}_t R} + j\Phi_{\tilde{s}_0\tilde{r}_t I}$$

$$\nabla_{\underline{w}_R} \mathrm{Re}\{\Phi_{\tilde{s}_0\tilde{r}_t}\underline{w}^*\} = \nabla_{\underline{w}_R} \mathrm{Re}\{(\Phi_{\tilde{s}_0\tilde{r}_t R} + j\Phi_{\tilde{s}_0\tilde{r}_t I})(\underline{w}_R - j\underline{w}_I)\}$$

$$= \nabla_{\underline{w}_R}\{\Phi_{\tilde{s}_0\tilde{r}_t R}\underline{w}_R + \Phi_{\tilde{s}_0\tilde{r}_t I}\underline{w}_I\}$$

$$= \Phi_{\tilde{s}_0\tilde{r}_t R}$$

Likewise,

$$\nabla_{w_I} \mathrm{Re}\left\{ \underline{\Phi}_{\tilde{s}_0\tilde{r}_t} w^* \right\} = \Phi_{\tilde{s}_0 \tilde{r}_{tI}}$$

$$\underline{w}^T \Phi_{\tilde{r}_t \tilde{r}_t} \underline{w}^* = \left(\underline{w}_R^T + j \underline{w}_I^T\right)\left(\Phi_{\tilde{r}_t\tilde{r}_{tR}} + j \Phi_{\tilde{r}_t\tilde{r}_{tI}}\right)\left(\underline{w}_R - j\underline{w}_I\right)$$

$$= \left(\underline{w}_R^T \Phi_{\tilde{r}_t\tilde{r}_{tR}} - \underline{w}_I^T \Phi_{\tilde{r}_t\tilde{r}_{tI}} + j\left(\underline{w}_R^T \Phi_{\tilde{r}_t\tilde{r}_{tI}} + \underline{w}_I^T \Phi_{\tilde{r}_t\tilde{r}_{tR}}\right)\right) \times \left(\underline{w}_R - j\underline{w}_I\right)$$

$$= \underline{w}_R^T \Phi_{\tilde{r}_t\tilde{r}_{tR}} \underline{w}_R - \underline{w}_I^T \Phi_{\tilde{r}_t\tilde{r}_{tI}} \underline{w}_R + \underline{w}_R^T \Phi_{\tilde{r}_t\tilde{r}_{tI}} \underline{w}_I + \underline{w}_I^T \Phi_{\tilde{r}_t\tilde{r}_{tR}} \underline{w}_I$$
$$+ j\left\{ \underline{w}_R^T \Phi_{\tilde{r}_t\tilde{r}_{tI}} \underline{w}_R + \underline{w}_I^T \Phi_{\tilde{r}_t\tilde{r}_{tR}} \underline{w}_R - \underline{w}_R^T \Phi_{\tilde{r}_t\tilde{r}_{tR}} \underline{w}_I + \underline{w}_I^T \Phi_{\tilde{r}_t\tilde{r}_{tI}} \underline{w}_I \right\}$$

But $\underline{w}^T \Phi_{\tilde{r}_t\tilde{r}_t} \underline{w}^*$ is real

and
$$\underline{w}_R^T \Phi_{\tilde{r}_t\tilde{r}_{tI}} \underline{w}_I = \left(\underline{w}_R^T \Phi_{\tilde{r}_t\tilde{r}_{tI}} \underline{w}_I\right)^T$$
$$= \underline{w}_I^T \Phi_{\tilde{r}_t\tilde{r}_{tI}}^T \underline{w}_R$$
$$= -\underline{w}_I^T \Phi_{\tilde{r}_t\tilde{r}_{tI}} \underline{w}_R$$

So

$$\underline{w}^T \Phi_{\tilde{r}_t\tilde{r}_t} \underline{w}^* = \underline{w}_R^T \Phi_{\tilde{r}_t\tilde{r}_{tR}} \underline{w}_R - \underline{w}_I^T \Phi_{\tilde{r}_t\tilde{r}_{tI}} \underline{w}_R + \underline{w}_R^T \Phi_{\tilde{r}_t\tilde{r}_{tI}} \underline{w}_I + \underline{w}_I^T \Phi_{\tilde{r}_t\tilde{r}_{tR}} \underline{w}_I$$

$$\Rightarrow \nabla_{\underline{w}_R} \underline{w}^T \Phi_{\tilde{r}_t\tilde{r}_t} w^* = 2\underline{w}_R^T \Phi_{\tilde{r}_t\tilde{r}_{tR}} - 2\underline{w}_I^T \Phi_{\tilde{r}_t\tilde{r}_{tI}}$$

$$\nabla_{\underline{w}_I} \underline{w}^T \Phi_{\tilde{r}_t\tilde{r}_t} w^* = 2\underline{w}_I^T \Phi_{\tilde{r}_t\tilde{r}_{tR}} + 2\underline{w}_R^T \Phi_{\tilde{r}_t\tilde{r}_{tI}}$$

Hence, combining the above

$$\nabla_{\underline{W}} R_e\{\Phi_{\tilde{\underline{s}}_o\hat{r}_t}\underline{W}^*\} = \nabla_{\underline{W}_R} R_e\{\Phi_{\tilde{\underline{s}}_o\hat{\underline{r}}_t}\underline{W}^*\} + j\nabla_{\underline{W}_I} R_e\{\Phi_{\tilde{\underline{s}}_o\hat{\underline{r}}_t}\underline{W}^*\}$$

$$= \Phi_{\tilde{\underline{s}}_o\hat{r}_{tR}} + j\Phi_{\tilde{\underline{s}}_o\hat{\underline{r}}_{tI}}$$

$$= \Phi_{\tilde{\underline{s}}_o\hat{r}_t}$$

$$\nabla_{\underline{W}}\underline{W}^T\Phi_{\hat{\underline{r}}_t\hat{\underline{r}}_t}\underline{W}^* = \nabla_{\underline{W}_R}\underline{W}^T\Phi_{\hat{\underline{r}}_t\hat{\underline{r}}_t}\underline{W}^* + j\nabla_{\underline{W}_I}\underline{W}^T\Phi_{\hat{\underline{r}}_t\hat{\underline{r}}_t}\underline{W}^*$$

$$= 2\underline{W}_R^T\Phi_{\hat{\underline{r}}_t\hat{\underline{r}}_{tR}} - 2\underline{W}_I^T\Phi_{\hat{\underline{r}}_t\hat{\underline{r}}_{tI}}$$

$$+j\left(2\underline{W}_I^T\Phi_{\hat{\underline{r}}_t\hat{\underline{r}}_{tR}} + 2\underline{W}_R^T\Phi_{\hat{\underline{r}}_t\hat{\underline{r}}_{tI}}\right)$$

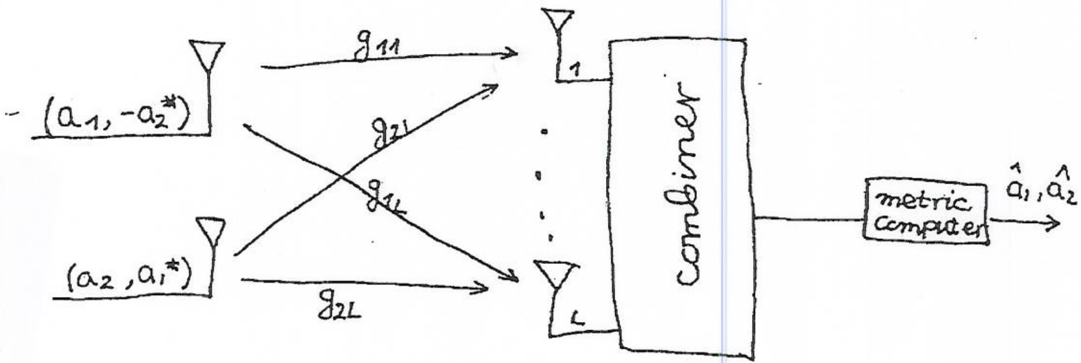
$$= 2\left(\underline{W}_R^T + j\underline{W}_I^T\right)\left(\Phi_{\hat{\underline{r}}_t\hat{\underline{r}}_{tR}} + j\Phi_{\hat{\underline{r}}_t\hat{\underline{r}}_{tI}}\right)$$

$$= 2\underline{W}^T\Phi_{\hat{\underline{r}}_t\hat{\underline{r}}_t}$$

Finally,

$$\nabla_{\underline{W}} J = 2\underline{W}^T\Phi_{\hat{\underline{r}}_t\hat{\underline{r}}_t} - 2\Phi_{\tilde{\underline{s}}_o\hat{\underline{r}}_t}$$

6.19 2×L Almaouti's combiner



g_{ij} - channel gain between transmit antenna i and receiver antenna j

$r_j^{(1)}$ – received signal at antenna j at time t

$r_j^{(2)}$ – received signal at antenna j at time $t+T$

The encoding sheme is:

- symbols a_1 and a_2 are transmitted from antennas 1 and 2 at time t

symbols $-a_2^*$ and a_1^* are transmitted from antennas 1 & 2 at time $t+T$

The received signals are

$$r_1^{(1)} = g_{11} a_1 + g_{21} a_2 + n_1^{(1)}$$
$$r_2^{(1)} = g_{12} a_1 + g_{22} a_2 + n_2^{(1)}$$
$$\vdots$$
$$r_L^{(1)} = g_{1L} a_1 + g_{2L} a_2 + n_L^{(1)}$$
$$r_1^{(2)} = g_{11}(-a_2^*) + g_{21} a_1^* + n_1^{(2)} \iff r_1^{(2)*} = -g_{11}^* a_2 + g_{21}^* a_1 + n_1^{(2)*}$$
$$\vdots$$
$$r_L^{(2)} = g_{1L}(-a_2^*) + g_{2L} a_1^* + n_L^{(2)} \iff r_L^{(2)*} = -g_{1L}^* a_2 + g_{2L}^* a_1 + n_L^{(2)*}$$

The combiner constructs two signal vectors:

$$V_1 = g_{11}^* \cdot r_1^{(1)} + g_{12}^* r_2^{(1)} + \dots + g_{1L}^* r_L^{(1)} + g_{21} \cdot r_1^{(2)*} + \dots + g_{2L} \cdot r_L^{(2)*}$$

$$V_2 = g_{21}^* r_1^{(1)} + g_{22}^* r_2^{(1)} + \dots + g_{2L}^* r_L^{(1)} - g_{11} r_1^{(2)*} - \dots - g_{1L} \cdot r_L^{(2)*}$$

6.20 We employ Alamouti's scheme on a <u>per subcarrier</u> basis. That is, treat each subcarrier independently.

Transmit OFDM symbol

	Period 1	Period 2
Ant 1	$\underline{X}_1$	$-\underline{X}_2^*$
Ant 2	$\underline{X}_2$	$\underline{X}_1^*$

$\underline{X}_1 = (X_{11}, X_{12}, \ldots X_{1N})$

$\underline{X}_2 = (X_{21}, X_{22}, \ldots X_{2N})$

X_{ij} = symbol from Ant i and subcarrier j